The word problem for free idempotent generated semigroups: An Italian symphony in $\mathsf{IG}(\mathcal{E})$ major

Igor Dolinka
Department of Mathematics and Informatics
University of Novi Sad
Trg Dositeja Obradovića 4, 21101 Novi Sad, Serbia
dockie@dmi.uns.ac.rs

Idempotent generated semigroups permeate semigroup theory, and, in fact, the entire combinatorial algebra. Hence, it would be very convenient to have at hand a sort of a "template" for an idempotent generated semigroup with a prescribed "structure" of idempotents. Luckily, such a "free-est" idempotent generated semigroup indeed exists, and it is defined with respect to a fixed *biordered set* $\mathcal{E}$, an ordered partial algebra that encapsulates the notion of a given "structure" of idempotents. This semigroup, called the *free idempotent generated semigroup on* $\mathcal{E}$ and denoted by $\mathsf{IG}(\mathcal{E})$, arises as an initial object in a suitably constructed category of semigroups and biordered set isomorphisms, but more importantly, it can be given by means of a presentation in which the generators are in a bijective correspondence with the elements of $\mathcal{E}$.

In the first part of my talk – that will be presented in a form akin to a cyclical musical piece, making a nod to Cremona's long and rich musical tradition – I will survey the main notions and early results/conjectures related to semigroups of the form $\mathsf{IG}(\mathcal{E})$. Then I will describe how the topic picked up pace significantly in the last 10 years, beginning with the ground-breaking 2009 paper of Brittenham, Margolis and Meakin which showed that the maximal subgroups of $\mathsf{IG}(\mathcal{E})$ are in fact fundamental groups of a suitably constructed cell 2-complex, and the subsequent discovery by Gray and Ruškuc that in fact, contrary to the previous belief, any (finitely generated) group arises as a maximal subgroup of $\mathsf{IG}(\mathcal{E})$ for a suitable (finite) $\mathcal{E}$. For some time, the focus was on further investigations of these maximal subgroups, but recently it switched to the study of the word problem. The speaker, Gray and Ruškuc jointly proved that the word problem for $\mathsf{IG}(\mathcal{E})$ (for finite $\mathcal{E}$) is always decidable when restricted to the regular part of this semigroup — provided that the maximal subgroups have decidable word problems — while there exists a finite biorder $\mathcal{E}$ such that $\mathsf{IG}(\mathcal{E})$ has an undecidable word problem, even though its maximal subgroups all have decidable word problems.

However, the main accent in the talk will be given to the recent joint paper of the speaker with Gould and Yang which translates the word problem of $\mathsf{IG}(\mathcal{E})$ completely into the realm of algorithmic problems in group theory. As it turns out, the word problem of $\mathsf{IG}(\mathcal{E})$ is actually equivalent to a family of specific constraint satisfaction problems (CSP) with maximal subgroups of $\mathsf{IG}(\mathcal{E})$ as domains and certain rational subsets of direct products of these subgroups as constraints. We shall discuss the main components of this result (which contains excerpts from combinatorics on words, automata, etc.), and demonstrate several far-reaching consequences.