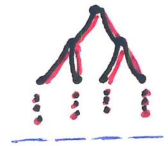
The Beautiful Simplicity of Flawless Vigorous Groups

 $\leftarrow \mathfrak{C}$!!

Joint with Luke Elliot & James Hyde.

Thm 1: Let $G \leqslant \mathrm{Aut}(\mathfrak{C})$. If G is vigorous or flawless then G is lawless.

Thm 3: Let G be vigorous, simple, and finitely generated, then G is 2-generated.

Thm 4: "Lots" of groups are vigorous and simple.

Thm 2: Let $G \leqslant \mathrm{Aut}(\mathfrak{C})$ be vigorous. T.F.A.E.

ⓐ G is simple.

ⓑ G is flawless.

ⓒ G is perfect & is generated by its elements of small support.

ⓓ G is perfect & approximately full.

ⓔ G is the commutator subgroup of its own full group in $\mathrm{Aut}(\mathfrak{C})$.

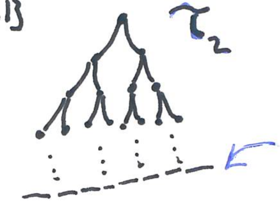
②

On Cantor Space

THM (Brouwer, 1910). Any two non-empty compact Hausdorff spaces without isolated pts and having countable bases consisting of clopen sets are homeomorphic to each other.

Ex: $\{0,1\}^{\mathbb{N}}$ $\quad V(\tau_2) = \{0,1\}^*$

τ_2



$\partial \tau_2 =: \mathcal{C}_2$

$:= \{ x_0 x_1 x_2 \cdots \mid x_i \in \{0,1\} \}.$

$u \in \{0,1\}^*$

$[u] = \{ uw \mid w \in \mathcal{C}_2 \} \quad \rightarrow$ Clopen!

- "Cone at u"

The cones form a basis for the topology of $\mathcal{C}_2$.

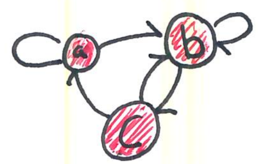
A set $U \in \mathcal{C}_2$ is $\boxed{\text{clopen}}$ $\Leftrightarrow$ it is a finite union of cones.

$$\boxed{K_{\mathcal{C}} = \text{The Proper clopen sets in } \mathcal{C}.}$$

1 (or 2) - sided $\boxed{\text{Path Space}}$

on this Graph ...

or



This graph

((sub)-shifts of finite type).

$$P(X) = \text{Path Space of } X := \left\{ f : \mathbb{Z} \to E(X) \;\middle|\; \begin{array}{c} t(f(i)) = s(f(i+1)) \\ \forall i \in \mathbb{Z} \end{array} \right\}.$$

NOTE The natural shift operator σ.

$\sigma : P(X) \to P(X)$.

$(x) \xmapsto{\sigma} (y) \quad (y_i = x_{i+1})$

On $G \leqslant \mathrm{Aut}(\mathbb{C})$.

$g \in \mathrm{Aut}(\mathbb{C}) \iff g$ is a Continuous Bijection $g : \mathbb{C} \to \mathbb{C}$.

— **Cool Fact:** $G = \langle X \mid R \rangle$ any countable group. Then $G \hookrightarrow \mathrm{Aut}(\mathbb{C})$.

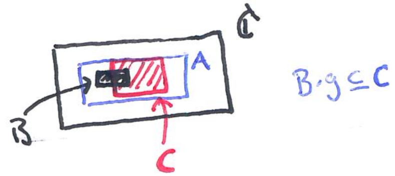
[– Fun exercise using $G \hookrightarrow \mathrm{Sym}(\mathbb{N})$]

— $g \in \mathrm{Aut}(\mathbb{C})$. $\mathrm{supt}(g) = \{ x \in \mathbb{C} \mid xg \neq x \}$ (right actions!).

— **Lemma:** $g, h \in \mathrm{Aut}(\mathbb{C})$. THEN $\mathrm{supt}(g^h) = \mathrm{supt}(g) \cdot h$.

$$\begin{cases} g^h = h^{-1} g h \\ [g,h] = g^{-1} h^{-1} g h \\ \quad = g^{-1} \cdot g^h \\ \quad = (h^{-1})^g \cdot h \end{cases}$$

Def: Let $G \leqslant \mathrm{Aut}(\mathbb{C})$. G is $\boxed{\text{vigorous}}$ $\iff$ $\forall A, B, C \in K_{\mathbb{C}}$ s.t. $B \cup C \subsetneq A$ $\exists g \in G$, $\mathrm{supt}(g) \subseteq A$, $B \cdot g \subseteq C$.



Let $W(x_1, \dots, x_j)$ be a valid expression. [redacted] $G \leqslant \mathrm{Aut}(\mathbb{C})$

$$X_{W,G} := \langle W(\alpha_1, \dots, \alpha_j) \mid \exists A \in K_{\mathbb{C}}, \forall i, \mathrm{supt}(\alpha_i) \subseteq \mathbb{C} \setminus A \rangle$$

Def: Let $\{1\} \neq G \leqslant \mathrm{Aut}(\mathbb{C})$, G is $\boxed{\text{Flawless}}$ $\iff$ $\forall$ valid expressions W, we have $\langle X_{W,G} \rangle = G$.

$\{x_1, x_2, \dots\}$ symbols.

$W(x_1, \dots, x_j)$ is a $\boxed{\text{valid expression}}$ if it is a freely reduced expression in variable symbols $\{x_1, x_2, \dots, x_j\} \cup \{x_1^{-1}, x_2^{-1}, \dots, x_j^{-1}\}$

e.g. $W(x_1, x_2) = x_1^{-1} x_2^{-1} x_1 x_2$

Given group values $\alpha_1, \dots, \alpha_j \in G$

$W(\alpha_1, \dots, \alpha_j)$ is the realisation of W at $(\alpha_1, \dots, \alpha_j)$.

— For a given valid expression W The set of all realisations of W in G generates The Verbal Subgroup G_W of G.

④

THM 1 (Almost trivial)

THM 1: Let $G \leqslant \mathrm{Aut}(\mathfrak{C})$. If G is Flawless or Vigorous Then G is Lawless.

Pf: Recall a Group is Lawless $\iff$ For any valid expression W, $\exists$ a realisation $W(\alpha_1, \dots, \alpha_j)$ That is non-trivial in G.

So By definition, if G is Flawless Then G is Lawless.

O.T.O.H. If G is Vigorous then

Path 1

Build a Ping-Pong.

But, Fully supported in a proper Clopen set!

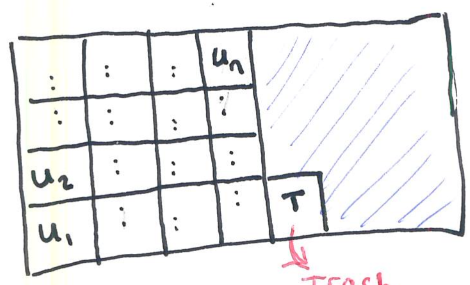
Easy.

$\Rightarrow$ G admits a non-abelian Free subgroup $\Rightarrow$ Lawless.

Path 2

For a given valid expression $W(\alpha_1, \dots, \alpha_j)$ use Vigorous property to explicitly build a non-trivial elt. realising W.

$\mathfrak{C}$



$|W| = n-1$

$W = y_1^{\varepsilon_1} y_2^{\varepsilon_2} \cdots y_{n-1}^{\varepsilon_{n-1}}$

each elt y_i moves u_i into u_{i+1}

Harder, but Fun.

Side Note: Flawless Groups must be perfect ($G = G'$). (Why?!)

$\therefore$ Flawless $\Rightarrow$ Perfect & Lawless

THM 2 ① ⇒ ②

Recall THM 2 ① ⇒ ② is the Statement: Let $G \leqslant Aut(\mathfrak{C})$ be vigorous and simple, then $\Longrightarrow$ G is Flawless.

Suppose G is vigorous & simple, and let $W(x_1, \dots, x_k)$ be a valid expression.

— By vigorous ⇒ Lawless proof $\exists$ proper clopen sets A, $B := \mathfrak{C} \setminus A$ and $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_k \in PStab_G(B)$ s.t. $\gamma := W(\alpha_1, \dots, \alpha_k)$ is not trivial.

— Now, $\forall g \in G$, $\gamma^g \in X_{G,W}$ as $\gamma^g = W(\alpha_1^g, \dots, \alpha_k^g)$, where $\forall i$ $\alpha_i^g \in PStab_G(B \cdot g)$.

— But by Simplicity of G

$$G = \langle \gamma^g \mid g \in G \rangle \leqslant \langle X_{G,W} \rangle.$$

And as W was an arbitrary valid expression we have G is Flawless !!

THM 2 ② ⇒ ①

The Converse (under General assumption of Vigorous) is: Suppose $G \leq \mathrm{Aut}(\mathfrak{C})$ is vigorous and flawless Then G is simple.

Suppose $G \leq \mathrm{Aut}(\mathfrak{C})$ is Vigorous & Flawless.

Let $\delta \in G \setminus \{1_G\}$. We will show the normal closure of δ is all of G.

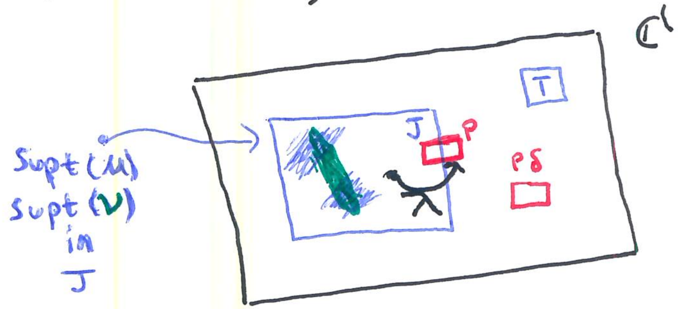
– Consider The valid expression $W(x_1, x_2) = [x_1, x_2] = x_1^{-1} x_2^{-1} x_1 x_2$. Note

$$X_{G, [x_1, x_2]} = \{ [\mu, \nu] \mid \exists J \text{ proper clopen}, \ \mu, \nu \in \mathrm{pStab}(\mathfrak{C} \setminus J) \}$$ Generates G.

– Let J proper clopen, and $\mu, \nu \in \mathrm{pStab}_G(\mathfrak{C} \setminus J)$. As $\delta \neq 1_G$ $\exists\, p \in K_{\mathfrak{C}}$, $p\delta \cap p = \emptyset$.
($J \in K_{\mathfrak{C}}$)

Let $\lambda \in G$ s.t. $J\lambda \subseteq p$. (By choice of p small, $J \cup p \cup p\delta \neq \mathfrak{C}$.)

- We can insist $\mathrm{supt}(\lambda) \subseteq J \cup p \cup T$



- $\mathrm{Supt}([\delta, \mu^\lambda]) \cap \mathrm{Supt}(\nu^\lambda) \leq p$.
- $[\delta, \mu^\lambda]$ [redacted] setwise stabilises p.
- ν^λ setwise stabilises p.
- $[\delta, \mu^\lambda]$ agrees with μ^λ on p

$$[\delta, \mu^\lambda] = \underbrace{((\mu^\lambda)^{-1})^{\delta}}_{\text{supt in } p\delta} \cdot \underbrace{\mu^\lambda}_{\text{supt in } p}$$

$$\Rightarrow \underbrace{[[\delta, \mu^\lambda], \nu^\lambda]}_{\text{in normal Closure of } \delta!}{}^{(\lambda^{-1})} = [\mu^\lambda, \nu^\lambda]^{(\lambda^{-1})} = [\mu, \lambda]. \Rightarrow \langle\langle \delta \rangle\rangle = G$$

(So, G is Simple!)

Full & Approximately Full

Let $G \leqslant \mathrm{Aut}(\mathcal{C})$.

— We say G is $\boxed{\text{Full}}$ $\Leftrightarrow$ whenever $K \in \mathbb{N}$ and

$$(\star)\begin{cases} \text{(a) } \{D_1, D_2, \dots, D_K\} \text{ and } \{R_1, \dots, R_K\} \text{ are partitions of } \mathcal{C} \text{ into } K \text{ proper clopen sets and} \\ \text{(b) } \{\gamma_1, \gamma_2, \dots, \gamma_K\} \subseteq G \text{ s.t. } \forall i,\ D_i \cdot \gamma_i = R_i \end{cases}$$

Then $\exists\, x \in G$ s.t. $\forall i$ $\quad x|_{D_i} = \gamma_i|_{D_i}$.

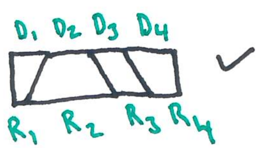
— We say G is Approximately Full $\Leftrightarrow$ whenever $K \in \mathbb{N}$ and $(\star)$

Then $\forall j \in \{1, 2, \dots, K\}$ $\exists\, X_j \in G$ s.t. $\forall i \neq j$ $\quad X_j|_{D_i} = \gamma_i|_{D_i}$

Full :

G is its own "piecewise Completion"

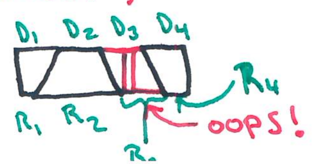
⤳ Build your own Frankenstein's Monster from all available pieces.



Approx. Full

G is almost it's own piecewise Completion

⤳ When you try to build your Frankenstein Monster, you get what you want except at some bit you choose, where something random happens!



THM 2 Recap

THM 3

<u>THM</u> Let $G \leq \mathrm{Aut}(\mathfrak{C})$ be Vigorous & Simple. If G is f.g. Then G is 2 generated by elts σ, τ, where you may specify $n \in \mathbb{N}$ so $|\sigma| = n$, and $|\tau| < \infty$.

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- Proof is hard & Technical.

⇝ Great reduction to complexity (compare w/Hyde's Thesis) when we ...

- introduced an abelian Group $(\mathcal{X}_G, +)$ (A Topological Invariant of G)

Similar to Matui's $H_0(\mathcal{G})$
Principle Etalé Groupoid.

- Motivated by Epstein 1970 → His argument <u>should</u> Give nice Finiteness properties.

It Probably Does!

THM 4

THM: "lots" of groups in $\mathrm{Aut}(\mathfrak{C})$ are (f.g.) vigorous & simple.

Let U be proper clopen in $\mathfrak{C}$. We say $G \leq \mathrm{Aut}(\mathfrak{C})$ is vigorous over U $\Leftrightarrow$

ⓐ $G \leq \mathrm{PStab}_{\mathrm{Aut}(\mathfrak{C})}(\mathfrak{C} \setminus U)$,

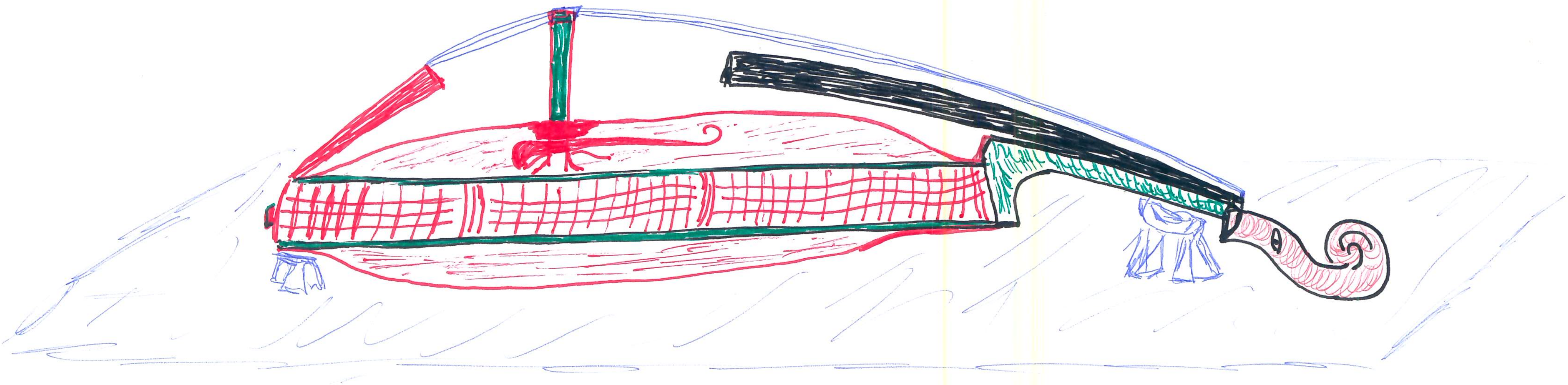
ⓑ $\forall$ A, B, C proper clopen with $B \cup C \subsetneq A \subseteq U$, $\exists \lambda \in G$, $\mathrm{supt}(\lambda) \subseteq A$, $B\lambda \subseteq C$.

THEN: ① If $G, H \leq \mathrm{Aut}(\mathfrak{C})$, U, V proper clopen in $\mathfrak{C}$, $U \cap V \neq \emptyset$, G vigorous over U, H vigorous over V, G, H simple $\Rightarrow$ $\langle G, H \rangle$ simple & vigorous over $U \cup V$.

② If $G \leq \mathrm{Aut}(\mathfrak{C})$, G simple & vigorous, P proper clopen with $\lambda \in G$, $P\lambda \cap P = \emptyset$. $\delta \in \mathrm{Aut}(\mathfrak{C})$, $\mathrm{Supt}(\delta) \subseteq P$ $\Rightarrow$ $\langle G, [\lambda, \delta] \rangle$ is simple & vigorous (and, still 2-gen if G was f.g.).

③ Many famous simple vigorous groups: $G'_{n,r}$, nV, $V(\Gamma)$, Many other Nekrashevych simple groups.

Grazie!



(Turn me, Please!)