Self-similar groups, automatic sequences, and unitriangular representations

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I introduction.

Finite automata

α) acceptors

↓

automatic groups
(e.g. hyperbolic groups)

β) transducers
(Mealy type)

↓

automaton groups
(self-similar groups)

↓

intermediate growth,
branch, torsion, ...

γ) automatically generated sequences

↓

topological full groups

↓

simple
amenable

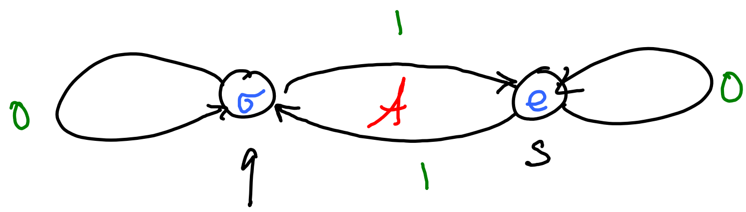
II Two examples.

$\{0,1\}$ – alphabet

$\{q,s\}$ – states

$\{e,\sigma\} = \mathrm{Sym}(2) \curvearrowright \{0,1\}$

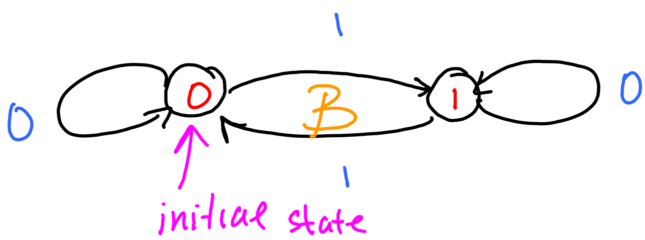
(i)



$$\mathcal{A}_q, \mathcal{A}_s : \left.\begin{array}{l} \{0,1\}^n \longrightarrow \{0,1\}^n \\ \{0,1\}^{\mathbb{N}} \longrightarrow \{0,1\}^{\mathbb{N}} \end{array}\right\} \text{ bijections}$$

$G = G(\mathcal{A}) = \langle \mathcal{A}_q, \mathcal{A}_s \rangle$ – group generated by $\mathcal{A}$ (automaton group)

$G \simeq \mathcal{L} = \mathbb{Z}/_{2\mathbb{Z}} \wr \mathbb{Z}$ – the Lamplighter group



$\mathcal{B}$ generates Thue-Morse sequence.

$$\eta = \{\eta_n\}_{n=0}^{\infty} = 011010011001\ldots\ldots$$

$$\mathbb{N} \ni n = i_0 + i_1 2 + \cdots + i_m 2^m$$

η_n is output of $\mathcal{B}$ after reading the word $i_m \cdots i_1 i_0$

Also η is fixed point of the map given by the substitution $\tau : 0 \longrightarrow 01, \quad 1 \longrightarrow 10$

$$0 \longrightarrow 01 \longrightarrow 0110 \longrightarrow 01101001 \longrightarrow \cdots$$

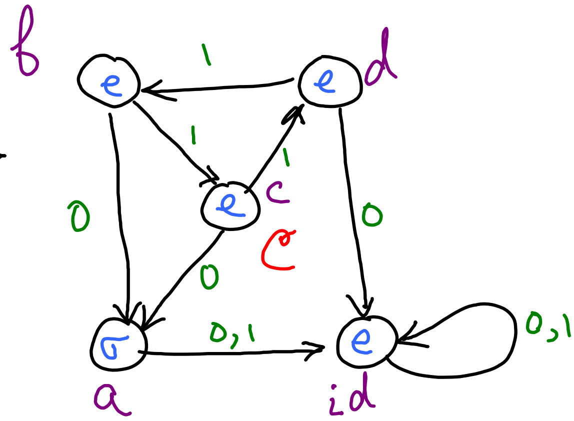
$$\eta = \lim_{n\to\infty} \tau^n(0)$$

(ii) $\mathcal{J} = G(\mathcal{C}) = \langle \mathcal{C}_a, \mathcal{C}_b, \mathcal{C}_c, \mathcal{C}_d \rangle$

$$= \langle a, b, c, d \mid 1 = a^2 = b^2 = c^2 = d^2 =$$

$$= bcd = \sigma^k((ad)^4) = \sigma^k((adacac)^4),$$

$$k = 0, 1, 2, \ldots \rangle$$

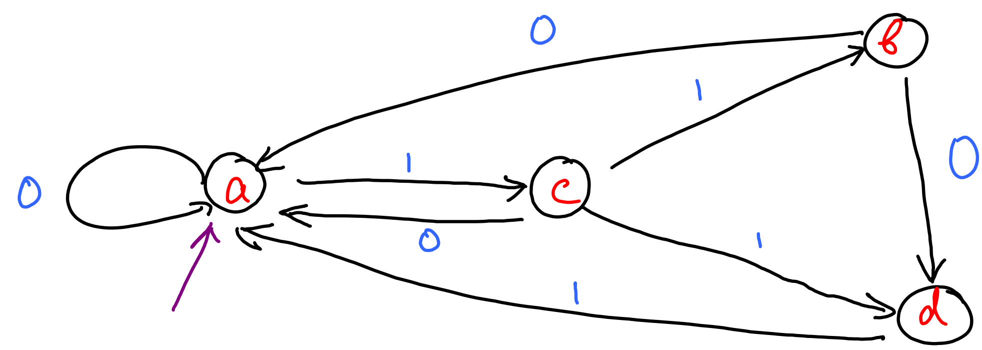


Lysenok presentation

$$\sigma: \begin{cases} a \longrightarrow aca \\ b \longrightarrow d \\ d \longrightarrow c \\ c \longrightarrow b \end{cases}$$

$$\xi = \lim_{n\to\infty} \sigma^n(a)$$

$$a \longrightarrow aca \longrightarrow acabaca \longrightarrow \cdots$$



automaton generating ξ

III Minimal Cantor systems

$A^{\mathbb{N}} \ni \omega \rightsquigarrow (\Omega, T)$

Ω is Cantor set

T shift

$\Omega \subset A^{\mathbb{Z}}$

Good sequences (like η, ξ) lead to minimal systems.

$$(\Omega, T)$$
$$\Downarrow$$
$$[[(\Omega, T)]] \leftarrow \text{amenable}$$

topological full group

amenable — Conjectured by Gr. & Medynets
Proved by Juschenko & Monod

Recent Nekrashevych results!
Matte-Bonn

$\mathcal{G} \hookrightarrow TFG$ generated by ξ Gr., Lenz, Nagnibeda

The subshift (Ω_ξ, T) associated with ξ leads to the information about spectra of Schreier graphs of $\mathcal{G}$ (via the relation to the random Schroedinger operator GLN)

IV Automatically generated sequences and matrices.

$a_n \in k$ - field of char p.

$\{a_n\}$ is p-automatic $\Leftrightarrow$ generated by a substitution

$$\Leftrightarrow \sum_{n=0}^{\infty} a_n z^n \text{ is algebraic over } k(x).$$

The notion generalizes to matrices (via the use of the double alphabet $A \times A$).

A matrix M is a column finite if the number of non zero entries in each column is finite.

$$M_\infty(k) \simeq \mathrm{End}\,(k^{\mathbb{N}})$$

$\mathcal{A}_d(k) \subset M_\infty(k)$

algebra of d-automatic (colum finite) matrices.

interested in representations

$G \longrightarrow M_\infty(k)$

$G \longrightarrow \mathcal{A}_d(k)$

$G \longrightarrow UT_\infty(k)$ – unitriangular representations

$G \longrightarrow$ Jacobi type representations

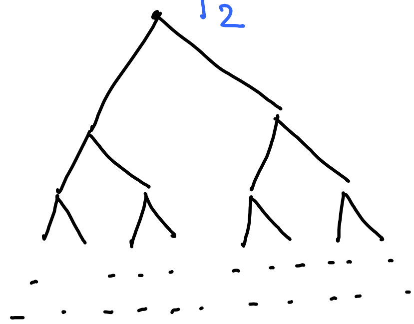
V Group actions on rooted trees

G residually finite group $\Leftrightarrow$ $G \hookrightarrow Aut(T_{\bar{m}})$

Spherically homogeneous rooted tree

T_d – d-regular rooted tree

T_2



$Aut(T_d)$ - profinite group

$\partial T = \{0,1\}^{\mathbb{N}}$ – Boundary

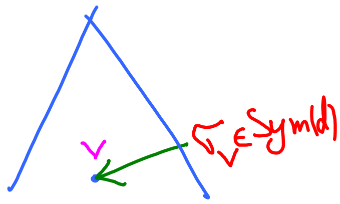
G - residually-p finite

$\longleftrightarrow$ $G \hookrightarrow Aut_{Syl}(T_p)$

X - alphabet $|X| = d \Rightarrow V(T_d) = X^* = \bigsqcup_{n=0}^{\infty} X^n$

$\mathrm{Aut}(T) \ni g \longleftrightarrow \mathcal{P}(g) \in \mathrm{Sym}(d)^V$

↑ portrait



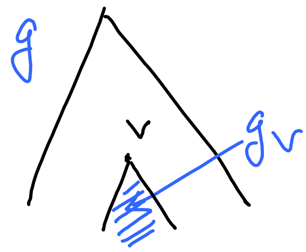
if $d = p$ - prime then

$\mathrm{Aut}_{syl}(T_p) = \{ g : \mathcal{P}(g) \in C^V \}$, $\quad C \simeq \mathbb{Z}/p\mathbb{Z}$ - cyclic

$C < \mathrm{Sym}(p)$

↑ Sylov p-subgroup of $\mathrm{Aut}(T_p)$.

g_v - section of g in vertex v.



Def. $\mathrm{Aut}(T_d) \ni g$ - finite state element if $\{g_v\}_{v \in V}$ is finite.

$\Longleftrightarrow$ g can be presented by finite state automaton

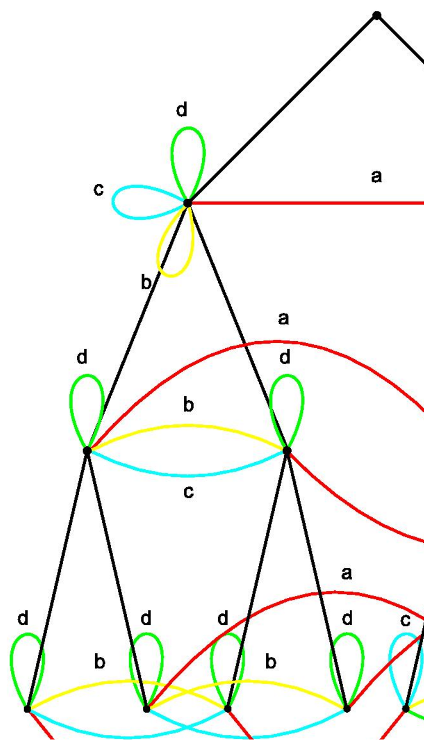
Many interesting groups are automaton groups and are subgroups of $\mathrm{Aut}_{syl}(T_p)$: G-groups, Gupta-Sidki groups, Basilica, ...

$G \curvearrowright \partial T = X^{\mathbb{N}}$ - boundary ($\approx$ Cantor set)

$(G, \partial T)$ minimal $\Leftrightarrow$ $G \curvearrowright T$ level transitive.

$\partial T \ni x \rightsquigarrow \Gamma_x$ - graph of action on the orbit of x.
(Schreier graph).

For $\mathcal{G} = \langle a, b, c, d \rangle \curvearrowright \partial T_2$ two types of graphs.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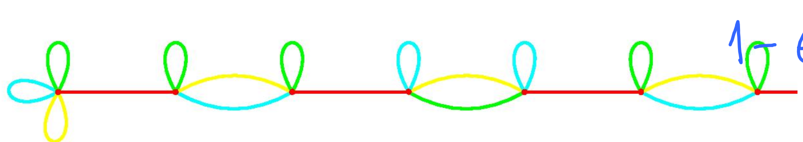
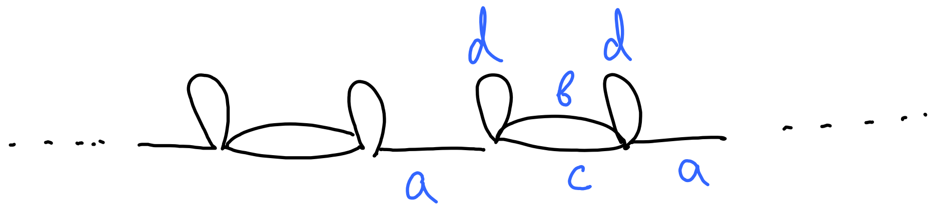
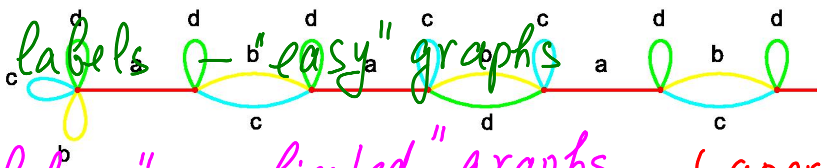


1-ended



2-ended

without labels – "easy" graphs



with labels "complicated" graphs (aperiodic order!)

The compact set $\{(\Gamma_x, x)\}_{x \in \partial T_2}$ is related to the subshift generated by sequence ξ. This reduces the spectral problem for Schreier graphs to the theory of Random Schroedinger operator.

Y. Vorobets, Gr. Lenz, Nagnibeda.

VI. Jacobi type representations.

$\Bbbk$ - field, V - infinite dimensional vector space over $\Bbbk$

$G = \langle S \rangle$ - finitely generated group.

Def. The representation $\pi : G \to GL(V)$ is said to be of *Jacobi type* (w.r. to S) if

$$\forall\, s \in S \qquad \pi(s) = \begin{pmatrix} \ddots & \ddots & & & 0 \\ \ddots & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & \ddots \end{pmatrix}$$

3-diagonal matrix.

1-sided
2-sided } versions of the definition

Def. Let k be a finite field. The representation $\pi : G \longrightarrow GL(V)$ is automatic representation if $\pi(s)$, $s \in S$ is automatic matrix ($\Leftrightarrow$ $\pi(g)$ is automatic for $\forall g \in G$).

Q. Which f.g. inf. groups have faithful Jacobi type representation?

Q Which groups have a faithful automatic representation?

Th 1. Let $k = \mathbb{C}$ or $\mathbb{F}_2$.

a) The group $\mathcal{G} = \langle a, b, c, d \rangle$ has faithful Jacobi type representations over k (both, 1-sided and 2-sided)

b) if $k = \mathbb{C}$ then $\mathcal{G}$ has $2^{\aleph_0}$ faithful Jacobi type 2-sided unitary irreducible representations. All of them are pairwise non equivalent.

c) $\mathcal{G}$ also has at least one 1-sided Jacobi type unitary irreducible representation. ($k = \mathbb{C}$)

d) if $k = \mathbb{F}_2$ then $\mathcal{G}$ has faithful 1-sided automatic Jacobi type representation. Moreover, the diagonals of $\pi(s)$, $s \in S = \{a, b, c, d\}$ are Toeplitz sequences.

[a sequence $\{a_n\}$ is called Toeplitz if $\forall n$ $\exists m$ such that $a_n = a_{n+im}$ for any $i = 1, 2, \ldots$]

VII Automatic representations

k - finite field, char $k = p$, X - alphabet

$V_n = k^{X^n}$ - vector space of functions $X^n \to k$

$T = T(X) = T_d$ if $|X| = d$ - rooted tree

$G \leq \mathrm{Aut}(T)$

$\pi_n : G \longrightarrow GL(V_n)$ - permutational representation of G in V_n coming from the action of G on X^n (nth level of T).

$$(\pi_n(g) f)(v) = f(g^{-1}v) \quad f \in V_n, \ v \in X^n.$$

V_n is G-module

$V_n \longrightarrow V_{n+1}$ – natural embedding (of G-modules)

$$V_\infty = \varinjlim_n V_n, \quad \pi_\infty = \varinjlim_n \pi_n$$

$C(X^{\mathbb{N}}, k)$ – vector space of continuous functions

boundary of the tree

$V_\infty \simeq C(X^{\mathbb{N}}, k)$ – isomorphism of G-modules

$(\pi_\infty(g) f)(w) = f(g^{-1} w)$, $w \in X^{\mathbb{N}}$

Basis: $\quad B$ – Basis in $V_1 = k^X$

$$B = \{ y_0, y_1, \dots, y_{d-1} \} \qquad y_0 < y_1 < \dots < y_{d-1}$$

$y_0 = \mathbb{1}$ – constant 1 function

order

$B^{\otimes n}$ – Basis in V_n

$B_\infty = \lim_{n \to \infty} B^{\otimes n}$ – Basis in $V_\infty = C(X^{\mathbb{N}}, k)$

B_∞ consist of functions of the form

$$f(x_0 x_1 \dots) = y_{i_0}(x_0) y_{i_1}(x_1) \dots .$$

$0 \le i_j \le d-1$, where all but a finite number of factors on the right-hand side are equal to the function $\mathbb{1}$.

Order B_∞ using the inverse lexicographic order.

$\boxed{\text{Th. 2}}$ The matrix of $\pi_\infty(g)$, $g \in G$ in the basis B_∞ is d-automatic $\Leftrightarrow$ g is a finite state.

$\boxed{\text{Cor.}}$ Automaton groups (over alphabet X, $|X|=d$) have a faithful automatic representation.

VIII

Unitriangular representations.

$|X| = p$ - prime, $X = \mathbb{F}_p$., $k = \mathbb{F}_p$

$$E = \{ \underset{\mathbb{1}}{\underset{\parallel}{e_0(x)}}, \underset{x}{\underset{\parallel}{e_1(x)}}, \underset{x^2}{\underset{\parallel}{e_2(x)}}, \dots, \underset{x^{p-1}}{\underset{\parallel}{e_{p-1}(x)}} \}$$ - basis in $V_1 = k^X$

Def. The basis E_∞ of $C(X^{\mathbb{N}}, \mathbb{F}_p)$ corresponding to E and consisting of all monomial functions $x_1^{i_1} x_2^{i_2} \dots$ on $X^{\mathbb{N}}$ ordered inverse lexicographically, so that

$$e_0 = \mathbb{1}, e_1 = x, \dots, e_{p-1} = x^{p-1}, e_p = x_2, e_{p+1} = x_1 x_2, \dots$$

is called Kaloujnine basis.

$e_n \in E_\infty \qquad e_n(x_1 x_2 \dots) = x_1^{i_1} x_2^{i_2} \dots$

where $i_1, i_2, \dots$ are the digits of the base p expension of n, i.e. such that $i_j \in \{0, 1, \dots, p-1\}$ and

$$n = i_1 + i_2 p + i_3 p^2 + \dots$$

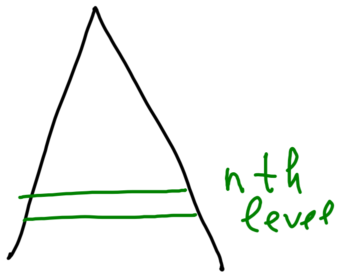
Th. 3. [Gr. Leonov, Nekrashevych, Suschansky]. The representation π_∞ of $\mathrm{Aut}_{sye}(T_p)$ in the Kaloujnine basis E_∞ is unitriangular. in particular, the representation π_∞ of all its subgroups $G \le \mathrm{Aut}_{sye}(T_p)$ are unitriangular. if G is automaton group then π_∞ is automatic representation.

One more basis in $V_1 = k^X$, $X = \mathbb{F}_p$

$$b_0 = 1, \quad b_k(x) = (-1)^k \binom{x+k}{k}, \quad k = 1, \dots, p-1$$

α : $\mathrm{Aut}_{Syl}(T_p) \longrightarrow \mathbb{F}_p^{\mathbb{N}}$ – abealization map

$$\alpha(g) = (\alpha_0(g), \alpha_1(g), \dots)$$



$\alpha_n(g) =$ the sum of elements of n-th level of portrait $\mathcal{P}(g)$

Th 4. [GLNS] a) Let $g \in Aut_{sye}(T_p)$ and let $A_g = (a_{ij})_{i,j=0}^{\infty}$ be the matrix of $\pi_\infty(g)$ in the above basis. Let $(s_1, s_2, \ldots) = (a_{01}, a_{12}, \ldots)$ be the first (above the main) diagonal. Then

$$s_n = \alpha_k(g),$$

where p^k is the maximal power of p dividing n.

b) if $g \in G < Aut_{sye}(T_p)$, where G is automaton group then $\{s_n\}$ is automatically generated Toeplitz sequence.

Example. $\mathcal{G} = \langle a, b, c, d \rangle$

$$\alpha(a) = (1, 0, 0, \dots)$$
$$\alpha(b) = (0, \underbrace{1, 1, 0}, 1, 1, 0, \dots)$$
$$\alpha(c) = (0, \underbrace{1, 0, 1}, 1, 0, 1, \dots)$$
$$\alpha(d) = (0, \underbrace{0, 1, 1}, 0, 1, 1, \dots$$

The first diagonal $\{S_n\}$ of $\pi_\infty(b)$ is given by:

$$S_{2^{3k}(2m+1)} = 0, \quad S_{2^{3k+1}(2m+1)} = 1, \quad S_{2^{3k+2}(2m+1)} = 1.$$

(Toeplitz sequence)

Generating functions $B_1(t)$, $C_1(t)$, $D_1(t)$ of the first diagonal of $\pi_\infty(b)$, $\pi_\infty(c)$, $\pi_\infty(d)$ satisfy

$$t^7 B_1^8 + B_1 + \frac{t^3}{1+t^8} + \frac{t}{1+t^4} = 0$$

$$t^7 C_1^8 + C_1 + \frac{t^7}{1+t^{16}} + \frac{t}{1+t^4} = 0$$

$$t^7 D_1^8 + D_1 + \frac{t^7}{1+t^{16}} + \frac{t^3}{1+t^8} = 0$$

For every $n \geq 1$ the generating functions $B_n(t)$, $C_n(t)$, $D_n(t)$ of n-th diagonals of $\pi_\infty(b)$, $\pi_\infty(c)$, $\pi_\infty(d)$ are of the form

$$\frac{P_0(t)}{1+t^{2^k}} + P_1(t)\, B_1(t)^{2^\ell}$$

where $k, \ell \geq 0$ are integers, and $P_0(t), P_1(t)$ are polynomials over $\mathbb{F}_2$.