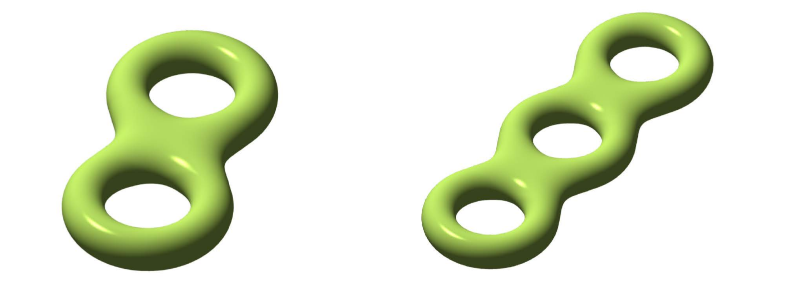
Automata in the study of surfaces groups

Tatiana Nagnibeda
University of Geneva

SandGAL, Cremona, June 12, 2019

FNSNF
Fonds national suisse de la recherche scientifique

This talk is devoted to a very classical class of groups: **the surface groups**. These are fundamental groups of oriented closed Riemannian surfaces Σ_g of genus g, $g \geq 2$.



Surface groups are finitely presented:

$$G_g = \langle a_1, ..., a_g, b_1, ..., b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle,$$

where $[a, b]$ denotes the commutator $aba^{-1}b^{-1}$.

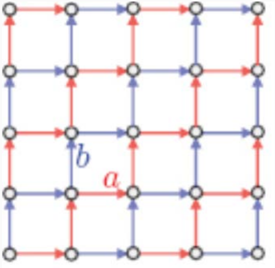
This is "the standard" presentation of G_g that comes from the Poincaré theorem, and we denote the standard (symmetric) system of generators

$$S_g = \{a_1^{\pm 1}, ..., a_g^{\pm 1}, b_1^{\pm 1}, ..., b_g^{\pm 1}\} .$$

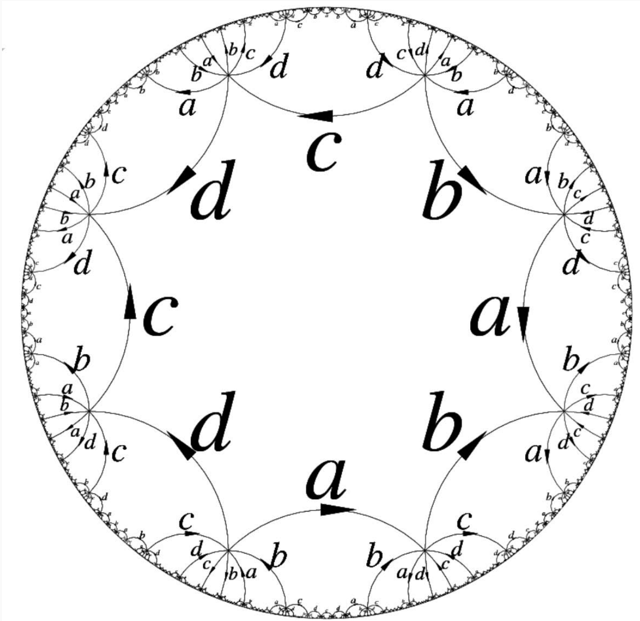
In the case of $g = 1$, the fundamental group of the torus is the free abelian group of rank 2

$$G_1 = \langle a, b \mid [a, b] = 1 \rangle.$$

G_1 is a cocompact lattice in the eucledian plane $\mathbb{R}^2$.



For $g \geq 2$, Σ_g has negative Euler characteristic, and G_g is a *cocompact lattice in the hyperbolic plane* $\mathbb{H}^2$.



Hence, the group G_g, $g \geq 2$ is *quasi-isometric* to $\mathbb{H}^2$. In particular, it is an example of a *word hyperbolic group*.

Recall that a **geodesic metric space is hyperbolic if all its triangles are thin**, in the following sense: there is a (global) constant $\delta \geq 0$ such that each edge of each triangle is contained in the δ-neighbourhood of the union of the other two edges of the triangle.

A finitely generated group is **word hyperbolic** (or Gromov hyperbolic) if it has a Cayley graph which is hyperbolic.

Hyperbolicity is invariant under quasi-isometry. Cayley graphs with respect to different finite generating sets are quasi-isometric, hence the property of being hyperbolic does not depend of the choice of a finite generating set in a group.

$\mathbb{Z}^2$ is not hyperbolic.

Word hyperbolic groups and in particular surface groups are examples of *automatic groups*.

Recall that a finitely generated group G given with a finite generating set S is *automatic* if there exists a normal form G (i.e. a language $L \subset S^*$ containing a unique representative of each element of G) which is regular, and such that two paths in the Cayley graph $Cay(G, S)$ corresponding to two elements in L and representing elements of G at distance 1 in $Cay(G, S)$ fellow-travel (i.e. remain at distance bounded by a constant $k > 0$, the same for all such pairs of paths).

Hyperbolic groups are automatic in a strong sense: the normal form in the definition above can be chosen to be geodesic, that is the length of a word in L coincides with the length of the element it represents in G.

Recall that *a language is regular if it is recognized by a finite state automaton.*

In his unpublished but influential 1980 paper *"The growth of the closed surface groups and the compact hyperbolic Coxeter groups"* Jim Cannon introduced the following definition:

Definition

The cone of a vertex x in $Cay(G, S)$ is the (rooted) subgraph spanned by the set of all vertices that can be connected to the identity element by a geodesic passing through x.

Two vertices x and y are said to have **the same cone type** if the graph automorphism taking x to y (given by the left multiplication by $x^{-1}y$ in G) takes the cone of x isomorphically onto the cone of y. In other words, a cone type in a Cayley graph $Cay(G, S)$ can be viewed as an orbit for the action of G on the set of cones of vertices in $Cay(G, S)$.

Cones can be considered with edges labelled by elements of $S \cup S^{-1}$. So we can talk about labelled or unlabelled cone types.

Theorem

([EPCHLT '92]) Any Cayley graph of any word hyperbolic group has finitely many cone types.

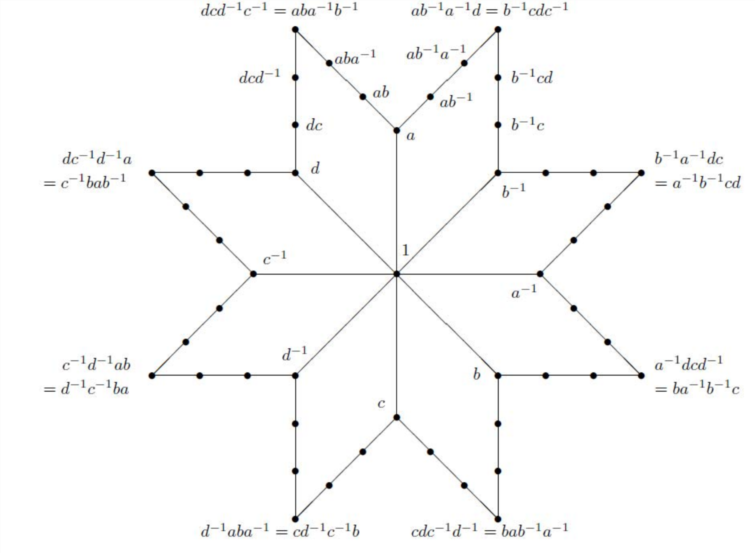
Coxeter groups are also known to have finitely many cone types in Cayley graphs with respect to Coxeter generators. In general, it is now known if having finitely many cone types is a quasi-isometric invariant; it may depend on the generating set.

Finitely many (labelled) cone types $\rightarrow$ **a finite state automaton of cone types, $\mathcal{A}_{ct}$.**

States of the automaton = cone types of vertices in $Cay(G, S)$. Directed edges = adjacency of cone types.

The automaton of labelled cone types recognizes the language of geodesics associated with (G, S).

Here is a zoom-in on the **Cayley graph** $Cay(G_2, S_2)$.



This graph is the 1-skeleton of the tesselation of $\mathbb{H}^2$ by regular $4g$-gons. It is $4g$-regular. It is planar self-dual. Hence there are $4g$ faces meeting at each vertex.

Interestingly, the Cayley graph $Cay(G_g, S_g)$, $g \geq 2$, is isomorphic to the Cayley graph of a certain Coxeter group

$$C_{4g} = \langle s_1, ..., s_{4g} \mid s_i^2 = 1, (s_i s_{i+1})^{2g} = 1, 1 \leq i \leq 4g \rangle.$$

(See Bozejko, Dykema and Lehner '06 for a combinatorial proof.)

Remark. There are $8g(2g-1)+1$ labelled cone types in $Cay(G_g, S_g)$ given by all subwords in the (cyclically written) relator $\prod_{i=1}^{g}[a_i, b_i]$ and its inverse.

There are $2g+1$ unlabelled cone types, you can view them on one side of a $4g$-gon at e.

Three applications of $\mathcal{A}_{ct}$ to the study of the group G

1. Growth. For a finitely generated group G with a finite system of semigroup generators $S = \{s_1, \dots, s_k\}, 1 \notin S$, the **growth series of** G with respect to S is the power series

$$f(z) \;=\; \sum_{g\in G} z^{|g|} \;=\; \sum_{n=0}^{\infty} a_n z^n \;\in\; \mathbb{Z}\big[[z]\big]\;,$$

where $|g|$ is the length of g in $S \cup S^{-1}$, and a_n is the number of elements of G of length n.

In [Grigorchuk-N '97] we also defined the **complete growth series** of G with respect to S as

$$F(z) \;=\; \sum_{g\in G} g z^{|g|} \;=\; \sum_{n=0}^{\infty} A_n z^n \;\in\; \mathbb{Z}[G]\big[[z]\big]\;,$$

where A_n is the sum of all elements of G of length n viewed as an element of the group ring: $A_n \;=\; \sum_{g\in G, |g|=n} g \quad \in\; \mathbb{Z}[G]$.

Theorem

If (G, S) has finitely many cone types, then its complete growth series is rational.

Rationality of the complete growth series $F(z)$ implies rationality of the growth series $f(z)$ via the augmentation map $g \mapsto 1$.

Virtually abelian groups have rational complete growth series (Liardet). Some of them have infinitely many cone types. Similarly, the growth series of the Heisenberg group H_3 is rational, but the number of cone types in the Cayley graph is infinite.

Proof: $F(z) = \sum_{i=1}^{K} F_i(z)$, where $F_i(z) = \sum_{n=0}^{\infty} A_n^i z^n$ with $A_n^i = \sum_{|g|=n, g \text{ of type } i} g$.

The coefficients A_n^i satisfy a **linear recursion**

$$(A_n^1, ..., A_n^K)^{tr} = \mathcal{R}(A_{n-1}^1, ..., A_{n-1}^K)^{tr}$$

with the matrix $\mathcal{R} = (r_{ij})_{1 \le i,j \le K}$ where $r_{ij} = succ(i, j)/pred(i)$.

The growth series of the surface group G_g, $g \geq 2$ with respect to the generators S_g was computed by Cannon:

$$f_g(z) = \frac{1 + 2z + 2z^2 + \ldots + 2z^{2g-1} + z^{2g}}{1 - (4g-2)z - (4g-2)z^2 - \ldots - (4g-2)z^{2g-1} + z^{2g}}.$$

It's denominator is a reciprocal **Salem polynomial** with exactly one root outside the unit disc.

This is also the case for some cocompact hyperbolic Coxeter groups.

One can rewrite this formula as

$$f_g(z) \;=\; \frac{(1+z)(1-z^{2g})}{1-(4g-1)z+(4g-1)z^{2g}-z^{2g+1}}.$$

Compare this with the formula for the complete growth series of G_g with respect to S_g:

$$F_g(z) \;=\; \frac{(1-z^2)(1-z^{4g})}{\Delta}\;, \qquad \text{with}$$

$$\Delta \;=\; 1-A_1 z+(4g-1)z^2+\mathcal{P}^b_{2g}z^{2g}-\mathcal{P}_{2g-1}z^{2g+1}+ \\ \mathcal{P}^b_{2g}z^{2g+2}+(4g-1)z^{4g}-A_1 z^{4g+1}+z^{4g+2},$$

a reciprocal polynomial with coefficients in $\mathbb{Z}[G_g]$ where $\mathcal{P}^b_{2g}$ is the sum of all subwords of length $2g$ with first letters in $\{b_1, b_1^{-1}, ..., b_g, b_g^{-1}\}$ of the cyclic words $r = \prod_{i=1}^{g}[a_i, b_i]$ and r^{-1}; and $\mathcal{P}_{2g-1}$ is the sum of all subwords of length $2g-1$ of the cyclic words r and r^{-1}.

2. Cogrowth and random walk. Suppose $G = F(S)/N$ where N is a normal subgroup in the free group $F(S)$. The **cogrowth series** of the pair G, S is defined as

$$l(z) \; = \sum_{g \in F(S): g \in N} z^{|g|} \; = \; \sum_{n=0}^{\infty} \lambda_n z^n \; ,$$

where λ_n is the number of loops without back-trackng based at e in the Cayley graph $Cay(G, S) =$ the number of freely reduced words of length n in $F(S)$ that are equal to 1 in G.

It turns out that there is an algebraic equation (discovered by Grigorchuk '78) that relates the cogrwoth series $l(z)$ and the generating series of the return probabilities for the simple random walk on $Cay(G, S)$

$$m(z) = \sum_{n=0}^{\infty} p^{(n)}(e, e) z^n.$$

In particular, the so-called spectral radius $\rho(G, S)$ of the random walk (which is the reciprocal of the radius of convergence of the series $m(z)$) is an algebraic number if and only if the rate of cogrowth of (G, S) is an algebraic number. *It is a long-standing open problem whether the spectral radius of the surface groups is an algebraic number.*

The language of freely reduced words in $F(S)$ that are equal to 1 in G is sometimes called WP, the "word problem" in (G, S). The word problem in a word hyperbolic group is solvable, i.e. there exists an algorithm that, given a word in S^* decides in finite time whether this word is equal to 1 in G or not. However *the complexity of WP as a language is not known even for surface groups.* The language WP is regular (which would imply that the cogrowth series is rational) iff the group is finite, and it is context free (whcih would imply that the cogrowth series is algebraic) iff the group is virtually free (Müller-Schupp).

It is not known whether the spectral radius of the surface group is an algebraic number, but we have good estimates on its value:

$$0.662772 \leq \rho(G_2, S_2) \leq 0.662816,$$

where $\rho(G,S) = \limsup_n p^{(2n)}(e,e)^{1/2n} \in [0,1]$ is the rate of decay of return probabilities for the simple random walk on the Cayley graph starting from e.

Numerical experiments suggest that the actual value is closer to the upper bound, obtained in [N, '97]. The lower bound was recently imroved to the above value in [Gouezel '15]. Both estimates are obtained with the help of the automaton $\mathcal{A}_{ct}$ of cone types.

The automaton $\mathcal{A}_{ct}$ being a finite directed graph, we can consider its directed cover: choose a starting point and construct a tree whose vertices are all the (directed) paths of finite length in $\mathcal{A}_{ct}$.

It turns out that this directed covering tree of the automaton $\mathcal{A}_{ct}$ admits still another interpretation: it can be understood as *the tree of geodesics* $T_{(G,S)}$ whose vertices are in one-to-one correspondence with the set of geodesics in $Cay(G,S)$ and two geodesics are connected by an edge iff one of them is an extension by one edge of the other.

The tree of geodesics of a Cayley graph with finitely many cone types is itself a graph with finitely many cone types. There is a natural projection of $T_{G,S}$ to $Cay(G,S)$:

$$\begin{array}{rccc} \theta: & V(T_{G,S}) & \to & G \\ & \gamma = [1_G, g] & \mapsto & g\ . \end{array}$$

This map is locally injective as the induced map on the set of edges is of the form $\big([1_G, g], [1_G, gs]\big) \mapsto (g, gs)$ (but not a local isomorphism). It preserves the cone types.

On the tree of geodesics we can consider the random walk that is the lift of the simple random walk on $\mathcal{A}_{ct}$. Its transition probabilities are determined by the cone types. *Random walks on trees with finitely many cone types* were studied in detail in [N, Woess '01]. In particular, the generating series of the return probabilities on such a tree is an algebraic function and $\rho(T_{(G,S)})$ is an algebraic number. It was shown in [N '04] that for (G, S) with finitely many cone types $\rho(G, S) \leq \rho(T_{(G,S)})$.

On the other hand, Gouezel estimated the spectral radius from below via *the Perron-Frobenius eigenvalue and the Perron-Frobenius eigenvector of the matrix $\mathcal{A}_{ct}$*.

Note that *the automaton $\mathcal{A}_{ct}$ for (G_g, S_g) is ergodic*, that is, the underlying graph is strongly connected, which allows application of the Perron-Frobenius thorem and guarantees the existence of the positive real eigenvalue biggest in absolute value, of multiplicity 1 and with positive eigenvector.

3. Unitary representations. Given a finitely generated group, it is in general a hard problem to classify its irreducible unitary representations. For n.e. hyperbolic groups there is no hope of a classification, but there are a few classes of representations that are known to come from various geometric constructions.

In a recent joint work [Kuhn, Manara, N. '19] we construct a family of irreducible unitary representations determined by the cone types automaton $\mathcal{A}_{ct}$.

In principle, the construction works for any group with finitely many cone types, under the condition that the cone type automaton is ergodic. However the technical details in the paper are done for the example of surface groups. Previously such representations were constructed by Kuhn and Steger '04 for free groups, but the construction there is considerably different, as the free group has infinite many ends, while the surface groups are one-ended.

Definition

A **matrix system** is given by the following data:

- a finite-dimensional complex vector space V_c, for each labelled cone type $c \in Vert(\mathcal{A}_{ct})$;
- a linear map $H_{c,c',s}\colon V_c \to V_{c'}$, for each directed edge $c \xrightarrow{s} c'$ in $\mathcal{A}_{ct}$;
- $H_{c,c',s} := 0$ fir all c a cone type and $s \in S$ a generator with $s \notin c$, and any c'.

The matrix system thus obtained is denoted $\{V_c, H_{c,c',s}\}$.

An important particular case is that of *scalar systems* where $V_c \simeq \mathbb{C}$ and every $H_{c,c',s}$ is represented by a complex scalar.

For a matrix system $\{V_c, H_{c,c',s}\}$ we define the finite dimensional vector space

$$V := \bigoplus_{c \in Vert(\mathcal{A}_{ct})} V_c.$$

For a geodesic path $\gamma = (v_0, ..., v_n)$ labelled by a word $s_1...s_n$ in the Cayley graph, we define the linear map $H_\gamma \colon V_{c_0} \to V_{c_n}$ as the composition

$$H_\gamma := H_{c_{n-1},c_n,s_n} \cdots H_{c_0,c_1,s_1}.$$

The map H_γ can be thought of as a map $H_\gamma \colon V \to V$.

Multiplicative functions are defined on the group G and take values in the vector space V.

Definition

Let $y \in G$ be a vertex of cone type c and let $v \in V_c$. The **elementary multiplicative function** supported on $Cone(y)$ with first value v is the function $m[y, v] \colon G \to V$ defined by

$$m[y,v](z) := \begin{cases} 0 & \text{if } z \notin Cone(y), \\ v & \text{if } z = y, \\ \sum_{\gamma\text{: a geodesic from } y \text{ to } z} H_\gamma v & \text{if } z \in Cone(y) \setminus \{y\}. \end{cases}$$

A **multiplicative function** is a linear combination of elementary multiplicative functions. The space of multiplicative functions is denoted by $\mathcal{H}^{\infty}(V_c, H_{c,c',s})$.

Our main technical result is that (on a surface group with a positive scalar system) there exists a positive definite sesquilinear form B on $V \otimes \overline{V}$ that induces a translation invariant inner product on $\mathcal{H}^{\infty}(V_c, H_{c,c',s})$, so that when we complete it, we get a *Hilbert space $\mathcal{H}_m(V_c, H_{c,c',s})$ of multiplicative functions*. The construction of B uses the Perron-Frobenius vector of an over matrix constructed from the cone type automaton and the scalar system.

For $g \in G$ and $f \in \mathcal{H}^{\infty}(V_c, H_{c,c',s})$ the left regular action of G on $\mathcal{H}_m(V_c, H_{c,c',s})$ is called a **multilplicative representation**:

$$(\pi_m(g)f)(z) := f\left(g^{-1}z\right), \quad z \in G.$$

Invariance of the inner product $\Rightarrow$ the representation is unitary.

Theorem

Any such multiplicative representation is tempered, i.e., is weakly contained in the left regular representation (the representation of G in the Hilbert space $l^2(G)$ with the left regular action).

Recall that *function of positive type* associated with a unitary representation π of a group G on a Hilbert space $\mathcal{H}$ is a function $g \in G \mapsto \langle \pi(g)\phi, \phi \rangle$, for a fixed vector $\phi \in \mathcal{H}$. Given unitary representations π_1, π_2, we say that π_1 is *weakly contained* in π_2 if all functions of positive type associated to π_1 can be approximated uniformly on the finite subsets of G by finite sums of functions of positive type associated to π_2.

For example, amenability of G is equivalent to the trivial representation being weakly contained in the left regular representation.

Any tempered representation gives rise to a *boundary representation* in the following sense.

Denote by ∂G *the Gromov boundary* of the Cayley graph $Cay(G, S)$ (i.e. the compactification of the metric space $Cay(G, S)$ defined as the space of equivalence classes of geodesic rays from e, where two geodesic rays are equivalent if they remain at bounded distance).

Any tempered representation can be extended to a representation of the cross-product $G \ltimes C(\partial G)$. We prove that it is irreducible.

Thank you!