New horizons in finite semigroup theory, 2019

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Mathematics + Finite Semigroups (Monoids)

Geometry (Topology) = Linked Equation

Mathematics = Presentation Lemma (PL)

Leibniz (1648 - 1716)

- Leibniz said "everything" can be written in $\{0,1\}^+$ and "everything" can be approximated as such.
- So restricting Geometry, Topology and Mathematics to Discrete Mathematics (thanks to D. Knuth and R. Graham) is *no* restriction.
- For example, restricting to finite topologies (not necessarily T_2 (= Hausdorff)) up to weak homotopy is the same as finite CW complexes up to homotopy (see the Introduction of
  - J. A. Barnak, Algebraic Topology of Finite Topological Spaces and Applications, Springer, 2011.
- "Really" finite posets, from a certain point of view, dating back to Hausdorff.

Linked equations

- The "linked equations", in finite semigroup theory of *partial* maps (partial maps is the "correct" choice).
- See the blackboard and:
  - I. Stein, The representation theory of the monoid of all partial functions on a set and related monoids as EI-category algebras, J. Algebra, 2016.
  - J. Rhodes and B. Steinberg, The q-theory of Finite Semigroups, Springer, 2009.
  - A. H. Clifford and G. B. Preston, The Algebraic Theory of Semigroups. Vol. I, AMS, 1961.

- Thus in the special case of GGM with each row having ≤ 2 1's (degree ≤ 2) and containing all rows with exactly one 1, we have the graph associated (b_1 —— b_2 if $b_1 \neq b_2$ and $c(b_1, a) = 1 = c(b_2, a)$ for some a).
- Then the automorphism group of the graph is the group of units of the TH (translational hull), and the partial map is linked if the inverse image of a point or edge is empty, a point or an edge.
- Thus partial maps (see I. Stein) between graphs (points), so f^{-1} of $x \in X = \{\emptyset\} \cup \{points\} \cup \{edges\}$ being another member of X is the "correct" definition of morphism.

See

- S. Margolis, J. Rhodes and P. V. Silva, On The Wilson Monoid of a Pairwise Balanced Design, preprint, 2019.
- J. H. Dinitz and S. Margolis, Continuous maps in finite projective space. In Proceedings of the thirteenth Southeastern conference on combinatorics, graph theory and computing, 1982.
- J. H. Dinitz and S. Margolis, Continuous maps on block designs, Ars Combin., 1982.
- R. M. Wilson, An existence theory for pairwise balanced designs i. composition theorems and morphisms, Journal of Comb. Th. (A), 1972.
- Current research.

Now on the blackboard I will talk about the first paper.

Stuart Margolis' later talk will be more detailed and inclusive.

Presentation Lemma

The PL in finite semigroups, a not necessarily decidable necessary and sufficient condition for $c(S) = n$. See:

- J. Rhodes and B. Steinberg, The q-theory of Finite Semigroups, Chapter 4.
- K. Henckell, J. Rhodes and B. Steinberg, An effective lower bound for group complexity of finite semigroups and automata, Trans. Amer. Math. Soc., 2012.

I cannot explain the PL in this time allowed (a semester might be enough!), so let me relate it to other mathematics, and show it is a fairly "standard" idea.

- Deciding c, based on extensive previous theory, comes down to

  Given a GM semigroup (S, A), is $Sc = RLMc$ or $RLMc + 1$?

- By Schützenberger representation $S \leq G \wr RLM$.
- PL "explains" when $Sc = RLMc$.
- Now GM is a G-bundle over RLM, see 100 pages on this in
  - S. Mac Lane and I. Moerdijk, Sheaves in geometry and logic, Springer-Verlag, 1994.
- This shows a major connection with standard mathematics (see blackboard).

Thank you for your interest and attention