Local finiteness for Green's relations in varieties of inverse semigroups and completely regular semigroups

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Joint work with

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Locally finite varieties of semigroups

- A semigroup variety is the class of all semigroups satisfying some collection of identities (like $xy = yx$ or $x^2 = x$)
- If such a collection can involve only finitely many variables, the variety is of finite axiomatic rank

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- If such a collection can involve only finitely many variables, the variety is of finite axiomatic rank
- A variety is locally finite if all its finitely generated members are finite
- For instance, the variety defined by $x^2 = x$ is locally finite while the variety defined by $xy = yx$ is not

Bounded Burnside problem

- A group G has finite exponent if it satisfies the identity $x^n = 1$ for some integer $n \geq 1$
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- The bounded Burnside problem asks if there exists an infinite finitely generated group with bounded exponent
- The positive answer was given in 1968 by Novikov and Adian
- We refer to infinite finitely generated groups of finite exponent as Novikov-Adian groups (NAGs)

Periodic varieties

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Periodic varieties

- A semigroup with zero S is a nilsemigroup if some power of each element in S is equal to zero
- A semigroup variety is periodic if all its one-generated members are finite
- Clearly, a locally finite variety must be periodic

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Theorem (Sapir 1987)

A periodic variety **V** of semigroups of finite axiomatic rank is locally finite if and only if:

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Theorem (Sapir 1987)

A periodic variety $\mathbf{V}$ of semigroups of finite axiomatic rank is locally finite if and only if:

(i) all nilsemigroups in $\mathbf{V}$ are locally finite;

(ii) $\mathbf{V}$ contains no NAGs.

Green's relations

The following five equivalence relations can be defined on every semigroup S:

- $x\mathcal{R}y$ if x and y are prefixes of each other
- $x\mathcal{L}y$ if x and y are suffixes of each other
- $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$
- $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$ $(= \mathcal{L} \circ \mathcal{R})$
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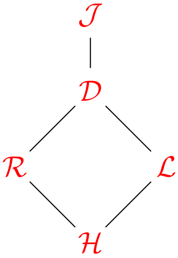
They were introduced by James Green in 1951 and are collectively referred to as Green's relations

Comparison

In general, we have:



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$$\begin{tikzcd} & \mathcal{J} \arrow[d, dash] & \\ & \mathcal{D} \arrow[dl, dash] \arrow[dr, dash] & \\ \mathcal{R} \arrow[dr, dash] & & \mathcal{L} \arrow[dl, dash] \\ & \mathcal{H} & \end{tikzcd}$$

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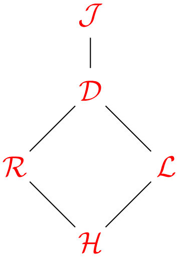

- Periodic semigroups: $\mathcal{J} = \mathcal{D}$
- Groups: all Green's relations coincide with the universal relation
- Nilsemigroups: all Green's relations coincide with equality

Locally Green finite varieties

- Let $\mathcal{K}$ be one of the five Green's relations
- A variety $\mathbf{V}$ of semigroups is said to be locally $\mathcal{K}$-finite if each finitely generated semigroup in $\mathbf{V}$ has only finitely many $\mathcal{K}$-classes

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- Volkov, Silva and Soares 2018: classification of locally $\mathcal{K}$-finite semigroup varieties

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- Motivation: a natural generalization of local finiteness that bypasses the classical Burnside problem (as all semigroup varieties consisting of groups are locally $\mathcal{K}$-finite for any $\mathcal{K}$)
- Of course, every locally finite variety is locally $\mathcal{K}$-finite for any $\mathcal{K}$; hence, only locally $\mathcal{K}$-finite varieties which are not locally finite are of interest

Not really...

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- $(\mathbb{N}, +)$ has infinitely many $\mathcal{J}$-classes
- Hence locally $\mathcal{K}$-finite varieties are periodic for every $\mathcal{K}$
- Every nilsemigroup is $\mathcal{J}$-trivial
- Thus, if $\mathbf{V}$ is a locally $\mathcal{K}$-finite variety, then nilsemigroups in $\mathbf{V}$ are locally finite
- Thus, Sapir's theorem tells us that every locally $\mathcal{K}$-finite variety of finite axiomatic rank which is not locally finite contains a NAG

Basic properties

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- Since locally $\mathcal{J}$-finite varieties must be periodic, a variety of semigroups is locally $\mathcal{J}$-finite if and only if it is locally $\mathcal{D}$-finite
- A variety of semigroups is locally $\mathcal{H}$-finite if and only if it is both locally $\mathcal{R}$-finite and locally $\mathcal{L}$-finite

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- Let $n \geq 665$ be odd and let $\mathbf{CR}_n$ consist of all semigroups which are unions of groups whose exponent divides n. Then $\mathbf{CR}_n$ is locally $\mathcal{D}$-finite but neither locally $\mathcal{R}$-finite nor locally $\mathcal{L}$-finite

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- The variety **LRO** of left regular orthogroups is locally $\mathcal{L}$-finite but not locally $\mathcal{R}$-finite

Results

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- We succeeded on characterizing all the locally $\mathcal{K}$-finite varieties of semigroups $\mathbf{V}$ for every Green's relation $\mathcal{K}$
- Forbidden objects are found for each one of the concepts
- Constructions involving NAGs are central in all these results

Results

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- A semigroup is completely regular if it is the union of its subgroups
- Some of the results involve reductions to the subvariety $\mathbf{CR(V)}$ containing all the completely regular semigroups in $\mathbf{V}$
- The cases of varieties of periodic completely regular semigroups and periodic semigroups with central idempotents were discussed in depth

$L_H(G)$

- Let G be a group and let H be a subgroup of G
- Denote by $L_H(G)$ the union of G with the set $G_H = \{gH \mid g \in G\}$ of the left cosets of H in G

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- Denote by $L_H(G)$ the union of G with the set $G_H = \{gH \mid g \in G\}$ of the left cosets of H in G
- Extend the multiplication in G to $L_H(G)$ by

$$g_1(g_2H) = g_1g_2H, \quad (g_1H)g_2 = (g_1H)(g_2H) = g_1H$$

Locally finite extensions

- Note that we view the coset gH as different from g even if H is the trivial subgroup
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- In the dual way, we define the semigroup $R_H(G) = G \cup \{Hg \mid g \in G\}$
- We say that a semigroup S is a locally finite extension of its ideal J if the Rees quotient S/J is locally finite

A theorem for $\mathcal{H}$

Theorem (VSS 2018)

A semigroup variety $\mathbf{V}$ of finite axiomatic rank is locally $\mathcal{H}$-finite if and only if either $\mathbf{V}$ is locally finite or satisfies the following conditions:

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A semigroup variety $\mathbf{V}$ of finite axiomatic rank is locally $\mathcal{H}$-finite if and only if either $\mathbf{V}$ is locally finite or satisfies the following conditions:

(i) every semigroup in $\mathbf{V}$ is a locally finite extension of a periodic completely regular ideal;

(ii) $\mathbf{V}$ contains none of the semigroups $L_H(G)$, $R_H(G)$, where G is a NAG and H is its subgroup of infinite index.

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- Since $(\mathbb{N}, +)$ is a subsemigroup of $(\mathbb{Z}, +)$ which is not a group, groups do not constitute a variety of semigroups
- But they constitute a variety of unary semigroups, where the unary operation is group inversion
- Similar problems affect other classes containing groups

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Varieties of unary semigroups

- Two important classes of semigroups (close to groups in different ways) constitute also varieties of unary semigroups:
  - the variety **CR** of completely regular semigroups
  - the variety **Inv** of inverse semigroups
- Both **CR** and **Inv** can be seen as I-varieties: subvarieties of the variety of I-semigroups (defined by the identities $x(yz) = (xy)z$, $(x')' = x$ and $xx'x = x$)

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- $S \in \mathbf{CR}$ is completely simple if $\mathcal{D}$ is the universal relation
- Completely simple semigroups can be described as Rees matrix semigroups

Basic facts

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- $\mathcal{J} = \mathcal{D}$ and is a congruence on every $S \in$ **CR**
- $S/\mathcal{D}$ is a semilattice and each $\mathcal{D}$-class is a completely simple semigroup
- But every semilattice is locally finite
- Hence **CR** is locally $\mathcal{D}$-finite (and consequently locally $\mathcal{J}$-finite)

$L_H(G)$ as a unary semigroup

- Let G be a group and let H be a subgroup of G
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- We write $L(G) = L_{\{1\}}(G)$
- Dually, we define $R(G)$

$\mathcal{R}$, $\mathcal{L}$ and $\mathcal{H}$

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Let $\mathbf{V}$ be an I-variety of completely regular semigroups.

(i) $\mathbf{V}$ is locally $\mathcal{R}$-finite if and only if $\mathbf{V}$ contains none of the completely regular semigroups $L(G)$ where G is either $\mathbb{Z}$ or a NAG.

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Inverse semigroups

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- A semigroup S is inverse if, for every $a \in S$, there exists a unique a^{-1} satisfying $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$
- Up to isomorphism, inverse semigroups are semigroups of partial injective functions closed under composition and inverse functions
- The unary operation is the obvious one

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In any variety of inverse semigroups:

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Adapting a construction of Norman Reilly

- Let G be a nontrivial group
- We define a semigroup with zero $\widetilde{N}(G) = G \cup (G \times G) \cup \{0\}$

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- Let G be a nontrivial group
- We define a semigroup with zero $\widetilde{N}(G) = G \cup (G \times G) \cup \{0\}$
- The multiplication in G is extended to $\widetilde{N}(G)$ by

$$g(h,k) = (gh,k), \quad (h,k)g = (h,kg),$$

$$(g,h)(k,\ell) = \begin{cases} (g,\ell) & \text{if } h = k \\ 0 & \text{if } h \neq k \end{cases}$$

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- If G is finitely generated, so is $\widetilde{N}(G)$
- $\widetilde{N}(G)$ has three $\mathcal{D}$-classes
- If G is infinite, $\widetilde{N}(G)$ has infinitely many $\mathcal{R}$-classes

Separating $\mathcal{D}$ from $\mathcal{R}$

Theorem (SSV 2019)

Let **V** be an *I*-variety of inverse semigroups containing some finitely generated $\mathcal{D}$-finite inverse semigroup which is not $\mathcal{R}$-finite.

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Let $\mathbf{V}$ be an I-variety of inverse semigroups containing some finitely generated $\mathcal{D}$-finite inverse semigroup which is not $\mathcal{R}$-finite. Then $\mathbf{V}$ contains the bicyclic monoid B or $\widetilde{N}(\mathbb{Z})$ or $\widetilde{N}(G)$ for some NAG G.

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The following conditions are equivalent for an I-variety $\mathbf{V}$ of inverse semigroups:

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Theorem (SSV 2019)

The following conditions are equivalent for an I-variety **V** of inverse semigroups:

(i) **V** is locally $\mathcal{H}$-finite;

(ii) **V** is locally $\mathcal{R}$-finite;

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(iv) **V** is locally $\mathcal{D}$-finite;

(v) **V** is locally $\mathcal{J}$-finite.

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Theorem (SSV 2019)

The following conditions are equivalent for an I-variety $\mathbf{V}$ of inverse semigroups:

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(ii) $\mathbf{V}$ is locally $\mathcal{R}$-finite;

(iii) $\mathbf{V}$ is locally $\mathcal{L}$-finite;

(iv) $\mathbf{V}$ is locally $\mathcal{D}$-finite;

(v) $\mathbf{V}$ is locally $\mathcal{J}$-finite.

Note that there exist locally $\mathcal{D}$-finite varieties which are not locally finite (e.g. the variety of groups)

We need something else

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- Let $\mathbf{W}_2$ be the variety of inverse semigroups defined by the identity $x^2 = 0$
- With the help of the infinite square-free word due to Morse and Hedlund, we can build a finitely generated $S \in \mathbf{W}_2$ with infinitely many $\mathcal{J}$-classes
- However, $\mathbf{W}_2$ does not contain neither B nor $\widetilde{N}(\mathbb{Z})$ nor $\widetilde{N}(G)$ for any NAG G

Uniform almost nilsemigroups

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Theorem (SSV 2019)

Let $\mathbf{V}$ be a variety of inverse semigroups. Then $\mathbf{V}$ is locally $\mathcal{D}$-finite if and only if all uniform almost nilsemigroups in $\mathbf{V}$ are locally finite.

Grazie! Thank you!