F-inverse monoids as algebraic structures in enriched signature

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Motivation

F-inverse monoid:
inverse monoid in which each σ-class contains a maximum element (w.r.t. $\leq$, σ — minimum group congruence)

> *max-element*: maximum element in a σ-class
>
> each F-inverse monoid is E-unitary, i.e., the idempotents form a σ-class

Kinyon (2014, 2018):

a natural unary operation on F-inverse monoids:

$$a \mapsto a^{\mathfrak{m}}, \text{ the maximum element of the } \sigma\text{-class } a\sigma$$

the class of F-inverse monoids — as algebras with the usual inverse monoid operations and this additional unary operation — forms a variety, i.e., can be defined by identities

Main topic

Natural problem: find a ‘nice’ model for the free object in this variety on any set X

Main topic of the talk:

to present such a model reminicent to the model of the free inverse monoid on X

more generally, to present an analogue of Margolis–Meakin expansion for F-inverse monoids

Introduction: X-generated algebraic structures

X — set
A — algebraic structure, briefly, algebra (of a given type τ)

inverse monoids (groups): algebras of type $(\cdot, {}^{-1}, 1)$

F-inverse monoids (groups) with unary operation ${}^{\mathfrak{m}}$:
algebras of type $(\cdot, {}^{-1}, {}^{\mathfrak{m}}, 1)$

$\mathbb{T}_X$ — term algebra (of type τ) on X

Basic properties of $\mathbb{T}_X$:

any mapping $f\colon X \to A$ uniquely extends to a morphism $\varphi\colon \mathbb{T}_X \to A$ (of algebras of type τ)

$t\varphi$ is the 'value' of $t \in \mathbb{T}_X$ in A, denoted $[t]_A$

the subalgebra of A generated by Xf is $\{[t]_A : t \in \mathbb{T}_X\}$

Introduction: X-generated algebraic structures

In special classes of algebras, terms are usually ‘simplified’.

For inverse monoids,

$\mathbb{T}_X$ is replaced by $\mathbb{I}_X$, the free monoid with involution on X

For F-inverse monoids with additional unary operation $^{\mathfrak{m}}$,

$\mathbb{T}_X$ will be also ‘simplified’ later on.

Introduction: X-generated algebraic structures

A, B — algebras (of type τ)
$\varphi\colon A \to B$ — morphism (of algebras of type τ)

A is an *X-generated algebra* if a mapping $i_A\colon X \to A$ is fixed s.t. Xi_A generates A

φ is a *canonical morphism* if it 'preserves the generators', i.e., $i_A\varphi = i_B$

Observations:

each canonical morphism is surjective

for any X-generated algebras A, B (of type τ), if there exists a canonical morphism $A \to B$ then it is uniquely determined

if A is an X-generated algebra (of type τ) and ρ is a congruence on A then A/ρ becomes X-generated w.r.t. $i_{A/\rho}\colon x \mapsto (xi_A)\rho$, and $\rho^\natural\colon A \to A/\rho$ becomes canonical

Introduction: Cayley graph of an X-generated group

G — X-generated group

Cayley graph Cay(G) of G:

vertices: $g \in G$

edges: $g \underset{x^{-1}}{\overset{x}{\rightleftarrows}} g[x]_G \quad (g \in G,\ x \in X)$

involution

G acts on Cay(G)

subgraph — edges are ‘in pairs’

e.g., $\Gamma_x : \quad 1 \underset{x^{-1}}{\overset{x}{\rightleftarrows}} [x]_G \quad (x \in X)$

$\Delta_g : \quad 1\bullet \qquad \bullet g \quad (g \in G)$

Introduction: Cayley graph of an X-generated group

paths in Cay(G), the category Cay$(G)^*$

empty paths: ε_g $(g \in G)$

the involution and the action of G extends to Cay$(G)^*$

the subgraph spanned by a path p: $\langle p \rangle$

Words and paths:

for every $g \in G$ and $w \in \mathbb{I}_X$, there is a unique path $p_g(w)$ in Cay(G) s.t. $\alpha p_g(w) = g$ and the label of $p_g(w)$ is w

Introduction: Margolis–Meakin expansion

G — X-generated group

Margolis–Meakin expansion $M(G)$ of G (Margolis–Meakin, 1989):

elements: (Γ, g) where Γ is a finite connected subgraph of $\mathrm{Cay}(G)$ with 1 and g as vertices

multiplication: $(\Gamma, g)(\Gamma', g') = (\Gamma \cup g\Gamma', gg')$

$M(G)$ is an X-generated E-unitary inverse monoid w.r.t.

$i_{M(G)} \colon x \mapsto (\Gamma_x, [x]_G)$, and the 2nd projection induces σ

Universal property of $M(G)$:

S — X-generated E-unitary inverse monoid

$\nu \colon G \to S/\sigma$ — canonical morphism

There is a canonical morphism $\varphi \colon M(G) \to S$ such that the diagram (of canonical morphisms) commutes.

$$\begin{tikzcd}
M(G) \arrow[r, dotted, "\varphi"] \arrow[d] & S \arrow[d] \\
G \arrow[r, "\nu"] & S/\sigma
\end{tikzcd}$$

Introduction: Margolis–Meakin expansion

Consequence:

F$\mathbf{G}_X$ — free X-generated group

$M(\mathrm{F}\mathbf{G}_X)$ is a free X-generated inverse monoid

φ is defined as follows:

$$(\Gamma, g)\varphi = [w]_S$$

for any $w \in \mathbb{I}_X$ s.t. $[w]_G = g$ and $\langle p_1(w)\rangle = \Gamma$

Main point of proof:

if $w, w' \in \mathbb{I}_X$ s.t. $[w]_G = [w']_G$ and $\langle p_1(w)\rangle = \langle p_1(w')\rangle$ then $[w]_S = [w']_S$

$M(G)$ as a machine: for any $w \in \mathbb{I}_X$, it computes $[w]_G$, all states accessed and all basic commands executed in each state

Introduction: Birget–Rhodes expansion

G — arbitrary group

Birget–Rhodes prefix expansion $\mathrm{BR}(G)$ *of* G
(Birget–Rhodes, 1984; Sz., 1989):

- elements: (A, g) where A is a finite subset of G containing 1 and g
- multiplication: $(A, g)(A', g') = (A \cup gA', gg')$

$\mathrm{BR}(G)$ is an F-inverse monoid with max-elements $(\{1, g\}, g)$, and the 2nd projection induces σ

Universal property of $\mathrm{BR}(G)$:

F — arbitrary F-inverse monoid
$\nu\colon\ G \to F/\sigma$ — arbitrary morphism

There is a unique morphism $\varphi\colon\ \mathrm{BR}(G) \to F$ *s.t. max-elements are mapped to max-elements and the diagram commutes.*

$$\begin{tikzcd}
\mathrm{BR}(G) \arrow[r, dotted, "\varphi"] \arrow[d] & F \arrow[d] \\
G \arrow[r, "\nu"] & F/\sigma
\end{tikzcd}$$

F-inverse monoids in enriched signature

From now on,
every F-inverse monoid is considered
as an algebra of type $(\cdot, {}^{-1}, {}^{\mathfrak{m}}, 1)$, and
F-inverse submonoids, morphisms and congruences of
F-inverse monoids
are understood in this context.

in particular,

any group is an F-inverse monoid where ${}^{\mathfrak{m}}$ is the identity mapping

for any inverse monoid which is F-inverse, σ is a congruence and $\sigma^{\natural}$ is a morphism of F-inverse monoids

the requirement 'max-elements are mapped to max-elements' for the morphism φ in the universal property of BR(G) says precisely that φ is a morphism of F-inverse monoids

F-inverse monoids in enriched signature

Kinyon (2014, 2018):

An algebra $(S; \cdot, ^{-1}, ^{\mathfrak{m}}, 1)$ is an F-inverse monoid if and only if $(S; \cdot, ^{-1}, 1)$ is an inverse monoid and the following holds:

$$a^{\mathfrak{m}} \geq a \text{ and } a^{\mathfrak{m}} = (ae)^{\mathfrak{m}} \text{ for all } a \in S \text{ and } e \in E(S).$$

Result:

The class of all F-inverse monoids forms a variety defined by any identity basis of the variety of inverse monoids together with the laws

$$x^{\mathfrak{m}} x^{-1} x = x \text{ and } (xy^{-1}y)^{\mathfrak{m}} = x^{\mathfrak{m}}.$$

Further identities valid in this variety:

$$(x^{\mathfrak{m}})^{-1} = (x^{-1})^{\mathfrak{m}} \quad \text{and}$$

$$(x_0 y_1^{\mathfrak{m}} x_1 \cdots x_{n-1} y_n^{\mathfrak{m}} x_n)^{\mathfrak{m}} = (x_0 y_1 x_1 \cdots x_{n-1} y_n x_n)^{\mathfrak{m}}.$$

F-inverse monoids in enriched signature

'Simplified' term algebra for F-inverse monoids:

the latter identities allow us to avoid $^{\mathfrak{m}}$-nested terms
terms without $^{\mathfrak{m}}$ can be simplified as for inverse monoids

$\mathbb{T}_X$ can be replaced by $\mathbb{Im}_X = (X \sqcup X^{-1} \sqcup \mathbb{I}_X^{\mathfrak{m}})^*$,
consisting of all terms of the form

$$t = u_0 v_1^{\mathfrak{m}} u_1 \cdots u_{n-1} v_n^{\mathfrak{m}} u_n$$

where $n \in \mathbb{N}_0$, $u_0, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{I}_X$, and
the unary operations $^{-1}$ and $^{\mathfrak{m}}$ are defined according to the identities above

Note: the empty symbol 1 is an identity element for $\cdot$ and $1^{-1} = 1$ but $1^{\mathfrak{m}} \neq 1$

The universal F-inverse monoid over an X-gen. group

Intuition: to generalise $M(G)$ for F-inverse monoids

we need a machine

amenable to terms $u_0 v_1^{\mathfrak{m}} u_1 \cdots u_{n-1} v_n^{\mathfrak{m}} u_n \in \mathbb{I}\mathfrak{m}_X$

outcome is 'the most general F-inverse monoid over G'

words $u_i, v_j \in \mathbb{I}_X$ should be interpreted as with $M(G)$

How to interpret the inputs of the form $v^{\mathfrak{m}}$?

for any X-generated F-inverse monoid F with $F/\sigma = G$,
we have $[v^{\mathfrak{m}}]_F = [\tilde{v}^{\mathfrak{m}}]_F$ provided $[v]_G = [\tilde{v}]_G$

thus machine should move from the latest state g to $g[v]_G$
without recording anything else

consequently,

(I) machine computes all (Γ, g) where Γ is a finite subgraph of $\mathrm{Cay}(G)$ with 1 and g as vertices

(II) besides moving along paths, 'jumping' between vertices should be allowed

(I) Definition of $F(G)$:

G — X-generated group

elements: (Γ, g) where Γ is a finite subgraph of $\mathrm{Cay}(G)$
with 1 and g as vertices

multiplication: $(\Gamma, g)(\Gamma', g') = (\Gamma \cup g\Gamma', gg')$

$F(G)$ is an X-generated F-inverse monoid w.r.t.
$i_{F(G)}\colon x \mapsto (\Gamma_x, [x]_G)$, the 2nd projection induces σ, and
$(\Gamma, g)^{\mathfrak{m}} = (\Delta_g, g)$

Remarks:

$M(G)$ is the inverse submonoid of $F(G)$ generated by $Xi_{F(G)} (= Xi_{M(G)})$
BR(G) is isomorphic to the inverse submonoid of $F(G)$ generated by $(F(G))^{\mathfrak{m}} = \{(\Delta_g, g) \colon g \in G\}$

Universal property of $F(G)$ — main result:

F — X-generated F-inverse monoid
$\nu \colon G \to F/\sigma$ — canonical morphism
There is a canonical morphism $\varphi \colon F(G) \to F$ such that the diagram (of canonical morphisms of F-inverse monoids) commutes.

$$\begin{tikzcd} F(G) \arrow[r, dotted, "\varphi"] \arrow[d] & F \arrow[d] \\ G \arrow[r, "\nu"] & F/\sigma \end{tikzcd}$$

(II) **Main concept of proof:**

journey in Cay(G):

non-empty sequence $(p_1, \ldots, p_n)$ of paths in Cay(G) where
 each p_i might be an also an empty path ε_g ($g \in G$)
 every sequence (p) is identified with p

journeys in Cay(G) form a category w.r.t. the *composition* defined by

$$(p_1, \ldots, p_m)(q_1, \ldots, q_n) = (p_1, \ldots, p_m q_1, \ldots, q_n)$$

provided $p_m\omega = \alpha q_1$

Note: $(p_1, \ldots, p_m q_1, \ldots, q_n) \neq (p_1, \ldots, p_m, q_1, \ldots, q_n)$

Fact: *every journey is a product of paths and 'jumps'* $(\varepsilon_g, \varepsilon_h)$

The universal F-inverse monoid over an X-gen. group

Terms and journeys:

to every $g \in G$ and $t \in \mathbb{Im}_X$, we assign a journey $j_g(t)$ in $\mathrm{Cay}(G)$ as follows: for any $w \in \mathbb{I}_X$,

$$j_g(w) = p_g(w) \quad \text{and} \quad j_g(w^{\mathfrak{m}}) = (\varepsilon_g, \varepsilon_{g[w]_G})$$

and, for any $u, v \in \mathbb{Im}_X$,

$$j_g(uv) = j_g(u) j_{g[u]_G}(v)$$

Note: assignment $t \mapsto j_1(t)$ is not injective, e.g.,

$$j_1(1^{\mathfrak{m}}) = j_1((xx^{-1})^{\mathfrak{m}}) = (\varepsilon_1, \varepsilon_1)$$

Main lemma:

if $t, t' \in \mathbb{I}\mathfrak{m}_X$ s.t. $[t]_G = [t']_G$ and $\langle j_1(t)\rangle = \langle j_1(t')\rangle$ then $[t]_F = [t']_F$

hence the universal property of $F(G)$ is straightforward with the following φ:

$$(\Gamma, g)\varphi = [t]_F$$

for any $t \in \mathbb{I}\mathfrak{m}_X$ s.t. $[t]_G = g$ and the journey $j_1(t)$ spans Γ

Corollary of the main result:

$F(\mathrm{F}\mathbf{G}_X)$ is a free X-generated F-inverse monoid

THANK YOU!