

F -inverse monoids as algebraic structures in enriched signature

Mária B. Szendrei

Bolyai Institute
University of Szeged

SandGAL 2019
Cremona, 10–13 June, 2019

joint work with Karl Auinger and Ganna Kudryavtseva

F-inverse monoid:

inverse monoid in which each σ -class contains a maximum element (w.r.t. \leq , σ — minimum group congruence)

max-element: maximum element in a σ -class

each *F*-inverse monoid is *E*-unitary, i.e., the idempotents form a σ -class

Kinyon (2014, 2018):

a natural unary operation on *F*-inverse monoids:

$a \mapsto a^m$, the maximum element of the σ -class $a\sigma$

the class of *F*-inverse monoids — as algebras with the usual inverse monoid operations and this additional unary operation — forms a variety, i.e., can be defined by identities

Natural problem: find a ‘nice’ model for the free object in this variety on any set X

Main topic of the talk:

to present such a model reminiscent to the model of the free inverse monoid on X

more generally, to present an analogue of Margolis–Meakin expansion for F -inverse monoids

Introduction: X -generated algebraic structures

X — set

A — algebraic structure, briefly, algebra (of a given type τ)

inverse monoids (groups): algebras of type $(\cdot, ^{-1}, 1)$

F -inverse monoids (groups) with unary operation m :
algebras of type $(\cdot, ^{-1}, ^m, 1)$

\mathbb{T}_X — term algebra (of type τ) on X

Basic properties of \mathbb{T}_X :

any mapping $f: X \rightarrow A$ uniquely extends to a morphism

$\varphi: \mathbb{T}_X \rightarrow A$ (of algebras of type τ)

$t\varphi$ is the ‘value’ of $t \in \mathbb{T}_X$ in A , denoted $[t]_A$

the subalgebra of A generated by Xf is $\{[t]_A : t \in \mathbb{T}_X\}$

Introduction: X -generated algebraic structures

In special classes of algebras, terms are usually ‘simplified’.

For inverse monoids,

\mathbb{T}_X is replaced by \mathbb{I}_X , the free monoid with involution on X

For F -inverse monoids with additional unary operation m ,

\mathbb{T}_X will be also ‘simplified’ later on.

Introduction: X -generated algebraic structures

A, B — algebras (of type τ)

$\varphi: A \rightarrow B$ — morphism (of algebras of type τ)

A is an X -generated algebra if a mapping $i_A: X \rightarrow A$ is fixed s.t. Xi_A generates A

φ is a *canonical morphism* if it ‘preserves the generators’, i.e.,
 $i_A\varphi = i_B$

Observations:

each canonical morphism is surjective

for any X -generated algebras A, B (of type τ), if there exists a canonical morphism $A \rightarrow B$ then it is uniquely determined

if A is an X -generated algebra (of type τ) and ρ is a congruence on A then A/ρ becomes X -generated w.r.t. $i_{A/\rho}: X \mapsto (xi_A)\rho$, and $\rho^\natural: A \rightarrow A/\rho$ becomes canonical

Introduction: Cayley graph of an X -generated group

G — X -generated group

Cayley graph $\text{Cay}(G)$ of G :

vertices: $g \in G$

edges: $g \begin{array}{c} \xrightarrow{x} \\ \xleftarrow{x^{-1}} \end{array} g[x]_G \quad (g \in G, x \in X)$

involution

G acts on $\text{Cay}(G)$

subgraph — edges are ‘in pairs’

e.g., $\Gamma_x: 1 \begin{array}{c} \xrightarrow{x} \\ \xleftarrow{x^{-1}} \end{array} [x]_G \quad (x \in X)$

$\Delta_g: 1 \bullet \quad \bullet g \quad (g \in G)$

Introduction: Cayley graph of an X -generated group

paths in $\text{Cay}(G)$, the category $\text{Cay}(G)^*$

empty paths: ε_g ($g \in G$)

the involution and the action of G extends to $\text{Cay}(G)^*$

the subgraph spanned by a path p : $\langle p \rangle$

Words and paths:

for every $g \in G$ and $w \in \mathbb{I}_X$, there is a unique path $p_g(w)$ in $\text{Cay}(G)$ s.t. $\alpha p_g(w) = g$ and the label of $p_g(w)$ is w

Introduction: Margolis–Meakin expansion

G — X -generated group

Margolis–Meakin expansion $M(G)$ of G (Margolis–Meakin, 1989):

elements: (Γ, g) where Γ is a finite connected subgraph of $\text{Cay}(G)$ with 1 and g as vertices

multiplication: $(\Gamma, g)(\Gamma', g') = (\Gamma \cup g\Gamma', gg')$

$M(G)$ is an X -generated E -unitary inverse monoid w.r.t.

$i_{M(G)}: x \mapsto (\Gamma_x, [x]_G)$, and the 2nd projection induces σ

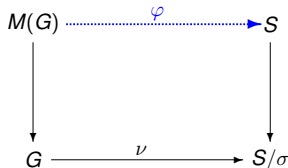
Universal property of $M(G)$:

S — X -generated E -unitary inverse monoid

$\nu: G \rightarrow S/\sigma$ — canonical morphism

There is a canonical morphism

$\varphi: M(G) \rightarrow S$ such that the diagram
(of canonical morphisms) commutes.



Introduction: Margolis–Meakin expansion

Consequence:

\mathbf{FG}_X — free X -generated group

$M(\mathbf{FG}_X)$ is a free X -generated inverse monoid

φ is defined as follows:

$$(\Gamma, g)\varphi = [w]_S$$

for any $w \in \mathbb{I}_X$ s.t. $[w]_G = g$ and $\langle p_1(w) \rangle = \Gamma$

Main point of proof:

if $w, w' \in \mathbb{I}_X$ s.t. $[w]_G = [w']_G$ and $\langle p_1(w) \rangle = \langle p_1(w') \rangle$ then
 $[w]_S = [w']_S$

$M(G)$ as a machine: for any $w \in \mathbb{I}_X$, it computes $[w]_G$, all states accessed and all basic commands executed in each state

Introduction: Birget–Rhodes expansion

G — arbitrary group

Birget–Rhodes prefix expansion $BR(G)$ of G

(Birget–Rhodes, 1984; Sz., 1989):

elements: (A, g) where A is a finite subset of G containing 1 and g

multiplication: $(A, g)(A', g') = (A \cup gA', gg')$

$BR(G)$ is an F -inverse monoid with max-elements $(\{1, g\}, g)$, and the 2nd projection induces σ

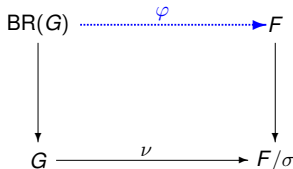
Universal property of $BR(G)$:

F — arbitrary F -inverse monoid

$\nu: G \rightarrow F/\sigma$ — arbitrary morphism

There is a unique morphism

$\varphi: BR(G) \rightarrow F$ s.t. max-elements are mapped to max-elements and the diagram commutes.



**From now on,
every F -inverse monoid is considered
as an algebra of type $(\cdot, ^{-1}, {}^m, 1)$, and
 F -inverse submonoids, morphisms and congruences of
 F -inverse monoids
are understood in this context.**

in particular,

any group is an F -inverse monoid where m is the identity mapping
for any inverse monoid which is F -inverse, σ is a congruence and
 σ^{\natural} is a morphism of F -inverse monoids

the requirement ‘max-elements are mapped to max-elements’ for
the morphism φ in the universal property of $\text{BR}(G)$ says precisely
that φ is a morphism of F -inverse monoids

Kinyon (2014, 2018):

An algebra $(S; \cdot, ^{-1}, {}^m, 1)$ is an F -inverse monoid if and only if $(S; \cdot, ^{-1}, 1)$ is an inverse monoid and the following holds:

$$a^m \geq a \text{ and } a^m = (ae)^m \text{ for all } a \in S \text{ and } e \in E(S).$$

Result:

The class of all F -inverse monoids forms a variety defined by any identity basis of the variety of inverse monoids together with the laws

$$x^m x^{-1} x = x \text{ and } (xy^{-1}y)^m = x^m.$$

Further identities valid in this variety:

$$(x^m)^{-1} = (x^{-1})^m \text{ and}$$

$$(x_0 y_1^m x_1 \cdots x_{n-1} y_n^m x_n)^m = (x_0 y_1 x_1 \cdots x_{n-1} y_n x_n)^m.$$

'Simplified' term algebra for F -inverse monoids:

the latter identities allow us to avoid m -nested terms
terms without m can be simplified as for inverse monoids

\mathbb{T}_X can be replaced by $\mathbb{I}m_X = (X \sqcup X^{-1} \sqcup \mathbb{I}_X^m)^*$,
consisting of all terms of the form

$$t = u_0 v_1^m u_1 \cdots u_{n-1} v_n^m u_n$$

where $n \in \mathbb{N}_0$, $u_0, \dots, u_n, v_1, \dots, v_n \in \mathbb{I}_X$, and
the unary operations $^{-1}$ and m are defined according
to the identities above

Note: the empty symbol 1 is an identity element for \cdot and
 $1^{-1} = 1$ but $1^m \neq 1$

The universal F -inverse monoid over an X -gen. group

Intuition: to generalise $M(G)$ for F -inverse monoids

we need a machine

amenable to terms $u_0 v_1^m u_1 \cdots u_{n-1} v_n^m u_n \in \mathbb{I}m_X$

outcome is ‘the most general F -inverse monoid over G ’

words $u_i, v_j \in \mathbb{I}_X$ should be interpreted as with $M(G)$

How to interpret the inputs of the form v^m ?

for any X -generated F -inverse monoid F with $F/\sigma = G$,

we have $[v^m]_F = [\tilde{v}^m]_F$ provided $[v]_G = [\tilde{v}]_G$

thus machine should move from the latest state g to $g[v]_G$

without recording anything else

The universal F -inverse monoid over an X -gen. group

consequently,

- (I) machine computes all (Γ, g) where Γ is a finite subgraph of $\text{Cay}(G)$ with 1 and g as vertices
- (II) besides moving along paths, 'jumping' between vertices should be allowed

(I) **Definition of $F(G)$:**

G — X -generated group

elements: (Γ, g) where Γ is a finite subgraph of $\text{Cay}(G)$
with 1 and g as vertices

multiplication: $(\Gamma, g)(\Gamma', g') = (\Gamma \cup g\Gamma', gg')$

$F(G)$ is an X -generated F -inverse monoid w.r.t.

$i_{F(G)}: x \mapsto (\Gamma_x, [x]_G)$, the 2nd projection induces σ , and

$$(\Gamma, g)^m = (\Delta_g, g)$$

The universal F -inverse monoid over an X -gen. group

Remarks:

$M(G)$ is the inverse submonoid of $F(G)$ generated by $Xi_{F(G)} (= Xi_{M(G)})$

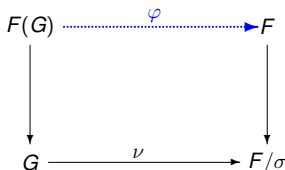
$BR(G)$ is isomorphic to the inverse submonoid of $F(G)$ generated by $(F(G))^m = \{(\Delta_g, g) : g \in G\}$

Universal property of $F(G)$ — main result:

F — X -generated F -inverse monoid

$\nu: G \rightarrow F/\sigma$ — canonical morphism

There is a canonical morphism $\varphi: F(G) \rightarrow F$ such that the diagram (of canonical morphisms of F -inverse monoids) commutes.



(II) Main concept of proof:

journey in $\text{Cay}(G)$:

non-empty sequence (p_1, \dots, p_n) of paths in $\text{Cay}(G)$ where
each p_i might be an also an empty path ε_g ($g \in G$)
every sequence (p) is identified with p

journeys in $\text{Cay}(G)$ form a category w.r.t. the *composition*
defined by

$$(p_1, \dots, p_m)(q_1, \dots, q_n) = (p_1, \dots, p_m q_1, \dots, q_n)$$

provided $p_m \omega = \alpha q_1$

Note: $(p_1, \dots, p_m q_1, \dots, q_n) \neq (p_1, \dots, p_m, q_1, \dots, q_n)$

Fact: every journey is a product of paths and 'jumps' ($\varepsilon_g, \varepsilon_h$)

The universal F -inverse monoid over an X -gen. group

Terms and journeys:

to every $g \in G$ and $t \in \mathbb{I}m_X$, we assign a journey $j_g(t)$ in $\text{Cay}(G)$ as follows: for any $w \in \mathbb{I}X$,

$$j_g(w) = p_g(w) \quad \text{and} \quad j_g(w^m) = (\varepsilon_g, \varepsilon_{g[w]_G})$$

and, for any $u, v \in \mathbb{I}m_X$,

$$j_g(uv) = j_g(u)j_{g[u]_G}(v)$$

Note: assignment $t \mapsto j_1(t)$ is not injective, e.g.,

$$j_1(1^m) = j_1((xx^{-1})^m) = (\varepsilon_1, \varepsilon_1)$$

Main lemma:

if $t, t' \in \mathbb{I}m_X$ s.t. $[t]_G = [t']_G$ and $\langle j_1(t) \rangle = \langle j_1(t') \rangle$ then $[t]_F = [t']_F$

hence the universal property of $F(G)$ is straightforward with the following φ :

$$(\Gamma, g)\varphi = [t]_F$$

for any $t \in \mathbb{I}m_X$ s.t. $[t]_G = g$ and the journey $j_1(t)$ spans Γ

Corollary of the main result:

$F(\mathbf{FG}_X)$ is a free X -generated F -inverse monoid

THANK YOU!