Quasi-ideal in Ternary semigroup

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Abstract

Sioson[9] in 1965 defined the left, right, lateral and quasi-ideal in ternary semigroup. Continuing that we discuss here the properties of ideals, quasi-ideals, minimal quasi-ideals and bi-ideals.

KEY WORDS AND PHRASES: Quasi-ideal, Bi-ideal, Ternary Semigroup.

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Introduction

D.H.Lehmer[6] gave the definition of ternary semigroup in 1932 as-

A non-empty set T is called a ternary semigroup if a ternary operation [ ] on T is defined and satisfies the associative law

$$[[x_1x_2x_3]x_4x_5] = [x_1\,[x_2x_3x_4\,]x_5] = [x_1x_2\,[x_3x_4x_5]]\ \forall\ x_i \in T\ \forall\ i, 1 \leq i \leq 5$$

Banach showed by an example that the ternary semigroup does not necessarily reduce to an ordinary semigroup.

While Los[7] showed that any ternary semigroup however may be embedded in an ordinary semigroup in such a way that the operation in the ternary semigroup is an (ternary) extension of the (binary) operation of the semigroup.

So many mathematicians Dudek[3], Feizullaer[4],Kim and Roush[5] and Sioson[9] did the work on ternary semigroup.

Quasi-ideal in Ternary semigroup

Sioson[9] gave the following definitions:

Definition 2.1: A left (right, lateral) ideal of a ternary semigroup T is a non-empty subset L(R,M) of T such that $[TTL] \subseteq L$ ($[RTT] \subseteq R$, $[TMT] \subseteq M$).

Definition 2.2: A non-empty subset of T is an ideal of T if it is left , lateral and right ideal of T.

Definition 2.3: For each element t in T , the left, right and lateral ideal generated by 't' are respectively given as:

- $(t)_L = \{t\} \cup [T\ T\ t]$
- $(t)_R = \{t\} \cup [t\ T\ T]$
- $(t)_M = \{t\} \cup [T\ t\ T] \cup [T\ T\ t\ T\ T]$.

contd.

Remark 2.5 : Every right, left and lateral ideal is a quasi-ideal. But every quasi-ideal is not a right, a left and a lateral ideal of T. This follows from the example.

Example 2.6 : $T = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ is a ternary semigroup under matrix multiplication. But

$Q = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ is a quasi-ideal of T , which is neither a left, nor a right nor a lateral ideal of T.

Definition 2.7: A ternary sub semigroup is a subset S of a ternary semigroup T such that [SSS]⊆S.

Definition 2.8 : A ternary semigroup T is a ternary group if

∀ x,y,z ∈ T ,∃ unique a,b,c in T such that [xab]=c, [ayb]=c and [abz]=c.

contd.

Definition 2.9 : A ternary semigroup T is with 0 if

$[000] = 0 = [0ab] = [a0b] = [ab0] = [00a] = [0b0] = [00c] \quad \forall a,b,c \in T.$

Definition 2.10 : An element a in T is idempotent if $[aaa] = a$.

Definition 2.11: A ternary semigroup T is with identity if there exists an idempotent element e in T such that $[aae] = a = [aea] = [eaa] \ \forall a \in T$.

Proposition 2.12: The intersection of a quasi-ideal Q and a ternary sub semigroup A of a ternary semigroup T is either empty or a quasi-ideal.

Proposition 2.13: Let Q be a non-empty subset of a ternary semigroup T, then the following are true:

(a) $Q \cup [TTQ]$ is the smallest left ideal of T containing Q.

(b) $Q \cup [QTT]$ is the smallest right ideal of T containing Q.

(c) $Q \cup [TQT] \cup [TTQTT]$ is the smallest lateral ideal of T containing Q.

(d) If Q is a quasi-ideal of T, then

$$Q = Q \cup \{ [TTQ] \cap ([TQT] \cup [TTQTT]) \cap [QTT]\}.$$

Proposition 2.14: Let X be a non- empty subset of a ternary semi group T, then

$(X)_q = (X\cup [TTX]) \cap (X\cup[TXT]\cup[TTXTT]) \cap (X\cup[XTT])$

$= X \cup \{ [TTX] \cap ([TXT] \cup [TTXTT]) \cap [XTT]\}$ is the smallest quasi-ideal containing X.

Bi-ideal in Ternary semigroup

Definition 3.1: A ternary sub semigroup B of a ternary semigroup T is a bi-ideal if $[BTBTB] \subseteq B$.

Proposition 3.2: Every quasi-ideal of a ternary semigroup T is a bi-ideal.

Proof: Let Q be a quasi-ideal of T. Then Q is a ternary sub semigroup of T. Now $[QTQTQ] \subseteq [Q[TTT]T] \subseteq [QTT]$.

Similarly, $[QTQTQ] \subseteq \{[TQT] \cup [TTQTT]\}$ and $[QTQTQ] \subseteq [TTQ]$.

Thus $[QTQTQ] \subseteq [QTT] \cap \{[TQT] \cup [TTQTT]\} \cap [TTQ] \subseteq Q$.

Proposition 3.3: Let A be an ideal and Q be a quasi-ideal of T , then $A \cap Q$ is a bi- as well as a quasi-ideal of T.

Proof: **[**$A \cap Q$ $A \cap Q$ $A \cap Q$] $\subseteq$[AAA]$\cap$[QQQ]$\subseteq A \cap Q$ implies $A \cap Q$ is a ternary sub semigroup of T. Also, **[**$A \cap Q$ T $A \cap Q$ T $A \cap Q$] $\subseteq$ **[**A [TAT]A] $\cap$ **[**QTQTQ] $\subseteq$ [AAA]$\cap$ Q$\subseteq$ $A \cap Q$.

Proposition 3.4: Let X and Y be non-empty subsets of a ternary semigroup T , then N=[XTY] is a bi-ideal of T.

Proposition 3.5: The intersection of an arbitrary set of bi-ideals of T is either empty or a bi-ideal of T.

Proposition 3.6: Every left, right or lateral ideal of T is a bi-ideal of T.

Proposition 3.7: Let Q be a subset of a ternary semigroup T and Y be a proper non-empty subset of T such that

(1) $[TTQ]\cup[TQT]\cup[QTT]\cup[TTQTT]\subseteq Y$

(2) $Y\subseteq Q$.

Then Y is an ideal of T. Moreover, Y is a bi-ideal of T.

The following example shows that both or either of the conditions (1) and (2) of above proposition are not satisfied then T is neither a left nor a right nor a lateral nor a quasi and nor a bi-ideal of T.

Example: Let T= $\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}\right\}$ be a ternary semigroup under multiplication.

Case1: If we take Y $=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\}$ and Q=$\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix}\right\}$, then Q is a quasi-ideal of T and

$[TTQ]\cup[TQT]\cup[QTT]\cup[TTQTT]=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}\right\}\nsubseteq Y$.

Neither of [TTY],[TYT],[TTYTT],[YTT] is in Y. So, Y is not an ideal of T.

$[TTY]\cap([TYT]\cup[TTYTT])\cap[YTT]\nsubseteq Y$. Thus, Y is not a quasi-ideal of T.

Case 2: Take Y=$\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right\}$ and Q=$\left\{\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right\}$implies Y⊆Q and [TTQ]∪[TQT]∪[QTT]∪[TTQTT]⊈ Y.

Since $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$∉ Y, so Y is not a quasi-ideal of T.

[YTYTY]⊈Y, Y is not a bi-ideal of T.

Case3: Take Y= $\left\{\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\}$and Q= $\left\{\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}$

[TTQ]∪[TQT]∪[QTT]∪[TTQTT]= $\left\{\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}$ ⊆ Y.

$\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$∈ [TTY], {[TYT]∪[TTYTT]}, [YTT] and $\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$∉Y.

So, Y is not an ideal of T.

Similarly, Y is neither a quasi nor a bi-ideal of T.

Proposition 3.8: Let X, Y and Z be three non-empty subsets of a ternary semigroup T and N=[XYZ]. Then N is a bi-ideal of T if one of the following conditions holds:

(1) X, Y $\subseteq$ Z and Z is a bi-ideal of T.

(2) Y, Z $\subseteq$ X and X is a bi-ideal of T.

(3) Z, X $\subseteq$ Y and Y is a bi-ideal of T.

(4) At least one of X,Y,Z is a right ,or a left, or a lateral ideal of T.

Definition 3.9 [9]: An element 't' in a ternary semigroup T is regular if there exists x,y in T such that [t x t y t] = t.

If all elements of T are regular, then T is said to be regular ternary semigroup.

We use definition 3.9 to show that a bi-ideal need not be a quasi-ideal.

Example 3.10: Let T be not a regular ternary semigroup and X,Y,Z be a minimal right, a minimal lateral and a minimal left ideal of T respectively. Then N=[XYZ] is a bi-ideal of T but N is not a quasi-ideal of T.

Proof: If possible, let N=[XYZ] be a quasi-ideal of T. $[XYZ]\subseteq[XTT]\subseteq X$, $[XYZ]\subseteq Y$, $[XYZ]\subseteq Z$. So $[XYZ]\subseteq X\cap Y\cap Z$ and $X\cap Y\cap Z$ is a minimal quasi-ideal [9] of T which implies $[XYZ]=X\cap Y\cap Z$, and hence T is a regular ternary semigroup [9]. So, our assumption is false.

Proposition 3.11: In a regular ternary semigroup, every bi-ideal is a quasi-ideal.

Proof: Sioson[9] shows that a subset Q of a regular ternary semigroup T is a quasi-ideal if and only if $[QTQTQ]\cap[QTTQTTQ]\subseteq Q$.

Proposition 3.12: Let C be a non-empty subset of a ternary semigroup T. Then C∪[CCC]∪[CTCTC] is the smallest bi-ideal of T containing C.

Proof: Let x ,y, z ∈ C∪[CCC]∪[CTCTC] be arbitrary elements.

x , y ,z ∈ C or x ,y ,z ∈ [CCC] or x , y, z ∈ [CTCTC]. Clearly,$[xt_9yt_{10}z]$ ∈ C∪[CCC]∪[CTCTC] for t_9 , t_{10}∈ T.

Hence C∪[CCC]∪[CTCTC] is a bi-ideal of T containing C.

Suppose there exists a bi-ideal R of T containing C such that

R⊆ C∪[CCC]∪[CTCTC] ⊆ R∪[RRR]∪[RTRTR]⊆ R.

R= C∪[CCC]∪[CTCTC] is the smallest bi-ideal containing C.

Minimal Quasi-ideal in Ternary semigroup

Here, we are concerning the ternary semigroup without 0 and has at least one idempotent. Henceforth every ternary semigroup with an idempotent element and without 0 is denoted by T.

Remark 4.1: If e is an idempotent element of T, then

(1) [eTT] is a minimal right ideal of T containing e.

(2) [TeT] is a minimal lateral ideal of T containing e.

(3) [TTe] is a minimal left ideal of T containing e.

Remark 4.2: If e is an idempotent element of T, then

$$[eTeTe] = [eTT] \cap [TeT] \cap [TTe].$$

Proposition 4.3: A quasi-ideal Q of T is minimal if and only if it is generated by any of its elements.

Proposition 4.4: A quasi-ideal Q of T is minimal if and only if Q is a ternary subgroup of T.

Proposition 4.5: Every minimal quasi-ideal Q of T can be written in the form Q=[eTeTe] where e is the identity of Q.

Proposition 4.6: Let e be an idempotent element contained in a minimal left ideal L (minimal lateral ideal M and minimal right ideal of R) of T, then [eeL]([eeMee], [Ree]) is a ternary subgroup of T.

Proof: According to Sioson[9] [eeL],[eeMee] and [Ree] are the quasi-ideals of T.

Since [Leeheeh] is the left ideal of T and [Leeheeh]$\subseteq$L for some h in L.

Therefore, minimality of L implies [Leeheeh]=L.

$\Rightarrow$[ee[Leeheeh]]=[eeL] for some h in L.

Thus, there exists a k in L such that [[eek][eeh][eeh]]=[eee]=e.

Hence[eeL] is a ternary subgroup and hence a minimal quasi-ideal of T.

Proposition 4.7: Let Q be a minimal quasi-ideal of T, then[eQs] and[sQe] are minimal quasi-ideals of T where s is any element of T and e is the identity element of Q.

Proposition 4.8: Let Q_1,Q_2 and Q_3 be the minimal quasi-ideals of T with identities e_1, e_2, e_3respectively. Then

$[e_1 Q_1 Q_2\, Q_3 e_3] = [\, e_1 TT] \cap [Te_2 T] \cap [TTe_3]$.

Proposition 4.9: Let Q_1 and Q_2 be the minimal quasi-ideals of T with identities e_1 and $e_2$respectively. Then

$[e_1 Te_1 Te_2 Te_2 Te_2 e_2 Te_1 e_1] = Q_1.$

$[e_2 Te_2 Te_1 Te_1 Te_1 e_1 Te_2 e_2] = Q_2.$

Now, we give simple but important results which we will use later.

Since e_1 is the identity of Q_1, therefore there exists s_1' , s_2', t_1' ,t_2' and t_3' in T such that

(A) $[e_1\ e_1\ [e_1 s_1'\ e_1]\ e_1\ [e_1 s_2'\ e_2] e_2 e_2 [t_1' e_2\ t_2' e_2 e_2] e_2 [e_2\ t_3' e_1] e_1] = e_1$.

Thus, $[e_1 e_1 s_1 e_1 s_2 e_2 e_2 t_1 e_2 t_2 e_1] = e_1$.

(B) Similarly, there exists t_1,t_2, s_1 and s_2 in T such that

$$[e_2 e_2 t_1 e_2 t_2 e_1 e_1 s_1 e_1 s_2 e_2] = e_2.$$

(C) Let e be the identity of T, then [ett]= t = [tet] = [tte] $\forall$ t $\in$ T

and [eet]= [ete]= [tee]= t $\forall$ t $\in$ T.

Proposition 4.10: All the minimal quasi-ideals of T are isomorphic.

Proof: Let Q_1 and Q_2 be the minimal quasi-ideals of T with identities e_1 and e_2 respectively.

Define a map ψ: $Q_1 \longrightarrow Q_2$ as

$\psi([e_1x_1e_1x_2e_1])=[e_2t_1e_2t_2e_1x_1e_1x_2e_1e_1s_1e_1s_2e_2e_2]$ $\forall$ $[e_1x_1e_1x_2e_1] \in Q_1$

where $t_1, t_2, x_1, x_2, s_1, s_2 \in$ T.

(A)$\Rightarrow$ there exists t_1,t_2, s_1 and s_2 in T such that $[e_1e_1s_1e_1s_2e_2e_2t_1e_2t_2e_1]= e_1$.

And $\psi[e_1x_1e_1x_2e_1] = \psi[e_1y_1e_1y_2e_1]$

$\Rightarrow[e_2t_1e_2t_2e_1x_1e_1x_2e_1e_1s_1e_1s_2e_2e_2] = [e_2t_1e_2t_2e_1y_1e_1y_2e_1e_1s_1e_1s_2e_2e_2]$

Applying $[e_1e_1s_1], [e_1s_2e_2]$on the left-hand side and $[t_1e_2t_2], e_1$on the right hand side under the ternary operation, we get

$[e_1x_1e_1x_2e_1]$ =$[e_1y_1e_1y_2e_1]$

Thus $\psi([[e_1x_1e_1x_2e_1][e_1y_1e_1y_2e_1][e_1z_1e_1z_2e_1]])$

=[$[e_2t_1e_2t_2e_1x_1e_1x_2e_1e_1s_1e_1s_2e_2e_2][e_2t_1e_2t_2e_1y_1e_1y_2e_1e_1s_1e_1s_2e_2e_2]$ $[e_2t_1e_2t_2e_1z_1e_1z_2e_1e_1s_1e_1s_2e_2e_2]$].

= [$\psi[e_1x_1e_1x_2e_1]$ $\psi[e_1y_1e_1y_2e_1]$ $\psi[e_1z_1e_1z_2e_1]$]. Thus, ψ is a homomorphism.

Define a map φ: $Q_2 \longrightarrow Q_1$ as φ ($[e_2x_1e_2x_2e_2]$)= $[e_1s_1e_1s_2\, e_2x_1e_2x_2e_2e_2t_1e_2t_2e_1e_1]$ $\forall$ $[e_2x_1e_2x_2e_2] \in Q_2$.

Then $\psi * \varphi = I_{Q_1}$ and $\varphi * \psi = I_{Q_2}$.

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Conclusion

A (binary) semigroup is a ternary semigroup but a ternary semigroup does not necessarily reduce to a (binary) semigroup.

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Thank You