Contributed talk in SandGAL 2019
Semigroups and Groups, Automata, Logics (10-13 June)

Epimorphically Preserved Heterotypical and Homotypical Semigroup Identities

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Outline

Abstract

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It is shown that a particular classes of Semigroup identities whose both sides contain repeated variables and are preserved under epis in conjunction with all seminormal identities.

Introduction and Preliminaries

Definition

A *semigroup identity* $u = v$, is the formal equality of two words u and v formed by letters over semigroup S.

Definition

A semigroup S is said to *satisfy an identity* if for every substitution of elements from S for the letters forming the words of the identity, the resulting words are equal in S.

Definition

The class of semigroups, in which a finite or an infinite collection $u_1 = v_1, \ u_2 = v_2, \ldots$ of identical relations is satisfied, is called the *variety of semigroups determined by these identical relations*, and the list of identical relations is called a *presentation of the variety*, denoted by $[u_1 = v_1, \ u_2 = v_2, \ldots]$.

Definition

Let u be any word. The *content* of u is the (necessarily finite) set of all the variables appearing in u, and will be denoted by $C(u)$. Further, for any variable x in u, $|x|_u$ will denote the number of occurences of the variable x in the word u.

Definition

An identity

$$u(x_1,\ x_2,\ldots,x_n) = v(x_1,\ x_2,\ldots,x_n),$$

in the variables $x_1, x_2, \ldots, x_n$ is called *homotypical* if $C(u) = C(v)$ and *heterotypical* otherwise.

Definition

An identity of the form

$$x_1 x_2 \cdots x_n = x_{i_1} x_{i_2} \cdots x_{i_n} \quad (n \geq 2) \tag{1}$$

is called a permutation identity, where i is any permutation of the set $\{1, 2, 3, \ldots, n\}$ and i_k $(1 \leq k \leq n)$ is the image of k under the permutation i. A permutation identity of the form (1) is said to be nontrivial if the permutation i is different from the identity permutation. Further a nontrivial permutation identity $x_1 x_2 \cdots x_n = x_{i_1} x_{i_2} \cdots x_{i_n}$ is called *left semicommutative* if $i_1 \neq 1$, *right semicommutative* if $i_n \neq n$ and *seminormal* if $i_1 = 1$ and $i_n = n$.

Remark

Every nontrivial permutation identity is either *left semicommutative*, *right semicommutative*, or *seminormal*.

Definition

A semigroup S satisfying a nontrivial permutation identity is said to be permutative, and a variety $\mathcal{V}$ of semigroups is said to be permutative if it admits a nontrivial permutation identity.

Remark

Commutativity $[xy = yx]$, left normality $[x_1x_2x_3 = x_1x_3x_2]$, right normality $[x_1x_2x_3 = x_2x_1x_3]$, and normality $[x_1x_2x_3x_4 = x_1x_3x_2x_4]$ are some of the well known permutation identities.

Definition

A morphism $\alpha : A \longrightarrow B$ in the category $\mathcal{C}$ of semigroups is said to be an *epimorphism* (*epi* for short) if for all morphisms $\beta,\ \gamma : B \longrightarrow C$, $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$.

Remark

One can easily see that any onto morphism is an epimorphism. Whether or not the converse is true, depends on the category under consideration. It is true in the categories of Sets, Abelian Groups and Groups for instance.
In general, epimorphisms are not onto in the categories of semigroups and rings.

Example

Take the embedding i of the real interval $(0,\ 1]$ into $(0,\ \infty]$, where both are considered as multiplicative semigroups. To see that $i : (0,\ 1] \longrightarrow (0,\ \infty]$ is epi, take any pair of homomorphisms α, β from $(0,\ \infty]$ such that $i\alpha = i\beta$; that is, α and β agree on $(0,\ 1]$. We shall show that for any $x > 1,\ x\alpha = x\beta$. Let $x > 1$. Then

$$[(x)\alpha(1/x)\alpha](x)\beta = (1)\alpha(x)\beta = (1)\beta(x)\beta = (x)\beta.$$

Equally though, since $1/x < 1$,

$$[(x)\alpha(1/x)\alpha](x)\beta = (x)\alpha[(1/x)\alpha(x)\beta] = (x)\alpha[(1/x)\beta(x)\beta]$$

$$= (x)\alpha(1)\beta = (x)\alpha(1)\alpha = (x)\alpha.$$

Therefore $\alpha = \beta$ and so i is epi.

Definition

Let U and S be any semigroups with U a subsemigroup of S. Following Isbell [8], we say that U dominates an element d of S if for every semigroup T and for all homomorphisms $\alpha, \beta : S \to T$, $u\alpha = u\beta$ for all $u \in U$ implies $d\alpha = d\beta$. The set of all elements of S dominated by U is called the *dominion* of U in S.

Definition

A semigroup U is said to be *epimorphically embedded* or *dense* in S if $Dom(U, S) = S$.

Remark

1. $Dom(U, S)$ is a subsemigroup of S containing U.
2. A morphism $\alpha : S \longrightarrow T$ is epi if and only if the inclusion $i : S\alpha \longrightarrow T$ is epi and the inclusion $i : U \longrightarrow S$ is epi if and only if $Dom(U, S) = S$.

Definition

A semigroup identity $u = v$ is said to be *epimorphically preserved* or *preserved under epis* if whenever S satisfies $u = v$ and $\alpha : S \longrightarrow T$ is epi, then T also satisfies $u = v$. Or equivalently, if whenever U satisfies $u = v$ and $Dom(U, S) = S$ implies that S also satisfies $u = v$.

An identity μ is said to be preserved under epis in conjunction with an identity τ if whenever S satisfies τ and μ, and $\varphi : S \to T$ is an epimorphism in the category of all semigroups, then T also satisfies τ and μ; or equivalently, whenever U satisfies τ and μ and $Dom(U, S) = S$, then S also satisfies τ and μ.

Since in category of semigroups epi need not be onto. So this idea focuses to study of semigroup identities preserved under epimorphism. Firstly, Isbell [10] showed that dominion of a commutative semigroup is commutative. Khan [11] showed that all semigroup identities are preserved under epis in conjunction with a commutative identity. But Higgins [6] showed that this result does not carry over to permutative semigroups i.e.the nontrivial permutation identities other than commutativity are not carried over to dominions. Afterwards, Khan [12] showed that all permutation identities are preserved under epis .

Moreover, Khan[13] also showed that all identities are preserved under epis in conjunction with a left(right) semicommutative permutation identity and all those identities whose atleast one side has no repeated variable are preserved under epis in conjunction with any nontrivial permutation identity. However, Higgins [6] had shown that the identity $xyx = yxy$ is not preserved under epis in conjunction with the normality identity $x_1x_2x_3x_4 = x_1x_3x_2x_4$ Thus the problem of finding those semigroup identities where both sides contain repeated variables and are preserved under epis in conjunction with a seminormal identity remains open. The authors Khan, Shah and Ashraf revisited this problem and found some sufficient conditions for certain classes of semigroup identities to be preserved under epis in conjunction with a seminormal identity.

A most useful characterization of semigroup dominions is provided by Isbell's Zigzag Theorem.

Result 2.1

[9, Theorem 2.3] Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows: $d = a_0 t_1 = y_1 a_1 t_1 = y_1 a_2 t_2 = y_2 a_3 t_2 = \cdots = y_m a_{2m-1} t_m = y_m a_{2m}$ (2) where $m \geq 1,\ a_i \in U\ (i = 0, 1, \ldots, 2m)$, $y_i, t_i \in S\ (i = 1, 2, \ldots, m)$, and $a_0 = y_1 a_1,\quad a_{2m-1} t_m = a_{2m},\quad a_{2i-1} t_i = a_{2i} t_{i+1},\quad y_i a_{2i} = y_{i+1} a_{2i+1}\quad (1 \leq i \leq m-1)$. Such a series of factorization is called a zigzag in S over U with value d, length m and spine $a_0, a_1, \ldots, a_{2m}$. We refer to the equations in Result 2.3 as the zigzag equations.

Result 2.2

[13, Theorem 3.1] All permutation identities are preserved under epis.

Result 2.3

[12, Result 3] Let U be any subsemigroup of a semigroup S and let $d \in Dom(U, S) \setminus U$. If (2) is a zigzag of minimal length m over U with value d, then $y_j, t_j \in S \setminus U$ for all $j = 1, 2, \ldots, m$.

Result 2.4

[12, Result 4] For any $d \in S \setminus U$ and k any positive integer, if (2) is a zigzag of minimal length over U with value d, then there exist $b_1, b_2, \ldots, b_k \in U$ and $d_k \in S \setminus U$ such that $d = b_1 b_2 \cdots b_k d_k$

Result 2.5

[12, Corollary 4.2] If U be permutative, then $sx_1 x_2 \cdots x_k t = sx_{j_1} x_{j_2} \cdots x_{j_k} t$, for all $x_1, x_2, \ldots, x_k \in S,\ s, t \in S \setminus U$ and any permutation j of the set $\{1, 2, \ldots, k\}$.

Result 2.6

[13, Proposition 4.6] Assume that U is permutative. If $d \in S \setminus U$ and (2) be a zigzag of length m over U with value d such that $y_1 \in S \setminus U$, then $d^k = a_0^k t_1^k$ for each positive integer k; in particular, the conclusion holds if (2) is of minimal length. Symmetrically, if $d \in S \setminus U$ and (2) be a zigzag of length m over U with value d such that $t_m \in S \setminus U$, then $d^k = y_m^k a_{2m}^k$ for each positive integer k; in particular, the conclusion holds if (2) is of minimal length.

Definition

Let S and Γ be any nonempty sets, then S is called Γ-semigroup if there exists a mapping from $S \times \Gamma \times S$ to S which maps (a, α, b) to $a\alpha b$ satisfying the condition $(a\alpha b)\gamma c = a\alpha(b\gamma c)$ for all $a, b, c \in S$ and for all $\alpha, \gamma \in \Gamma$.

Definition

Let S_1 be Γ_1 semigroup and S_2 be Γ_2 semigroup. A pair of mappings $f_1 : S_1 \to S_2$ and $f_2 : \Gamma_1 \to \Gamma_2$ is said to be homomorphism from (S_1, Γ_1) to (S_2, Γ_2) if $f_1(s_1\gamma s_2) = f_1(s_1)f_2(\gamma)f_1(s_2)$ for all $s_1 \in S_1, s_2 \in S_2$ and $\gamma \in \Gamma_1$.

Definition

Let U be nonempty subset of Γ-semigroup S, then U is called Γ-subsemigroup of S if $U\Gamma U \subset S$. A Γ-semigroup is called Γ-monoid if there exists $1 \in S$ such that $1\gamma s = s = s\gamma 1$ for all $s \in S$.Similarily, we can define Γ-submonoid.

Example

Let S be the set of all non-positive integers and Γ be the set of all non-positive even integers. If $a\alpha b$ denote as usual multiplication of integers for all $a, b \in S$ and all $\alpha \in \Gamma$,then S is a Γ-semigroup.

Definition

Let U be Γ-subsemigroup of S, then we say that U dominates an element d of S if for every Γ-semigroup T and for all homomorphisms $\alpha, \beta : S \to T$ and $\alpha^{'}, \beta^{'} : \Gamma \to \Gamma^{'}$, $u\alpha = u\beta$ and $\gamma\alpha^{'} = \gamma\beta^{'}$ for all $u \in U$ implies $d\alpha = d\beta$. The set of all elements of S dominated by U is called the *dominion* of U in S, and we denote it by $\Gamma - Dom(U, S)$.

Main results

Theorem

All semigroup identities of the form $x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} = w_1^{t_1} w_2^{t_2} \cdots w_n^{t_n} z_1^{l_1} z_2^{l_2} \cdots z_m^{l_m}$; *are preserved under epis in conjunction with all seminormal permutation identities for all non negative integers* $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s, t_1, t_2, \ldots, t_n, l_1, l_2, \ldots, l_m$ $(r, s, n, m \geq 1)$; *where* $p_1 \leq p_2 \leq \cdots \leq p_{r-1} \leq p_r$, $q_s \leq q_{s-1} \cdots \leq q_2 \leq q_1$, $t_1 \leq t_2 \leq \cdots \leq t_n$ *and* $l_m \leq l_{m-1} \cdots \leq l_2 \leq l_1$.

Theorem

All semigroup identities of the form:
$x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} = 0$ *for all positive integers* $p_1, p_2, \ldots, p_r$; *where* $p_1 \leq p_2 \leq \cdots \leq p_{r-1} \leq p_r$, $q_s \leq q_{s-1} \leq \cdots \leq q_2 \leq q_1$; *are preserved under epis in conjunction with all seminormal permutation identities.*

Theorem

All semigroup identities of the forms:

(*i*) $z^q x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} = x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r}; (p_r \leq p_{r-1} \cdots \leq p_2 \leq p_1, q \geq 0).$ ..... *(18)*

(*ii*) $x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} z^q = x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r}; (p_1 \leq p_2 \leq \cdots \leq p_{r-1} \leq p_r, q \geq 0).$ ..... *(19)*

are preserved under epis in conjunction with all seminormal permutation identities.

Theorem

Let u and v are any words in the variables $z_1, z_2, \ldots, z_\ell$ *such that* $|z_i|_u = |z_i|_v$ *for all* $i = 1, 2, \ldots, \ell$. *Then all semigroup identities of the form* $x_1^{p_1} \cdots x_r^{p_r} u(z_1, \ldots, z_\ell) y_1^{q_1} \cdots y_s^{q_s} = x_1^{p_1} \cdots x_r^{p_r} v(z_1, \ldots, z_\ell) y_1^{q_1} \cdots y_s^{q_s}$ $(\ell \geq 3)$ . . . . (8) *are preserved under epis in conjunction with (1), where* $\mathbf{p_1}, \ldots, \mathbf{p_r}, \mathbf{q_1}, \ldots, \mathbf{q_s}$ *will stand for any positive integers such that* $\mathbf{p_1} + \cdots + \mathbf{p_r} \geq \mathbf{g_0} - \mathbf{1}, \mathbf{q_1} + \cdots + \mathbf{q_s} \geq \mathbf{h_0} - \mathbf{1}$, $\mathbf{p_1} \leq \cdots \leq \mathbf{p_{r-1}} \leq p_r$ *and* $\mathbf{q_s} \leq \cdots \leq \mathbf{q_2} \leq \mathbf{q_1} (\mathbf{r}, \mathbf{s} \geq \mathbf{1})$.
Where $g_0 = \min P$, *the minimum of P, where* $P = \{2 \leq g \leq n-2 : x_{g-1}x_g$ *is not a subword of* $x_{i_1} x_{i_2} \cdots x_{i_n}\}$; *and let* $h_0 = \min Q$, *where* $Q = \{1 \leq h \leq n - g_0 - 1 : x_{n-h}x_{n-(h-1)}$ *is not a subword of* $x_{i_1} x_{i_2} \cdots x_{i_n}\}$.

Theorem

Let u and v be any words in the variables $z_1, \ldots, z_\ell$. If $\min\{|z_i|_u, |z_i|_v\} \geq \max\{p_r, q_1\}$ for all i in $\{1, 2, \ldots, \ell\}$, then all semigroup identities of the form

$$x_1^{p_1} \cdots x_r^{p_r} u(z_1, \ldots, z_\ell) y_1^{q_1} \cdots y_s^{q_s} = x_1^{p_1} \cdots x_r^{p_r} v(z_1, \ldots, z_\ell) y_1^{q_1} \cdots y_s^{q_s} \ (\ell \geq 3), \quad \ldots . (12)$$

are preserved under epis in conjunction with (1), where $\mathbf{p_1}, \ldots, \mathbf{p_r}, \mathbf{q_1}, \ldots, \mathbf{q_s}$ will stand for any positive integers such that $\mathbf{p_1} + \cdots + \mathbf{p_r} \geq \mathbf{g_0} - \mathbf{1}, \mathbf{q_1} + \cdots + \mathbf{q_s} \geq \mathbf{h_0} - \mathbf{1}$, $\mathbf{p_1} \leq \cdots \leq \mathbf{p_{r-1}} \leq p_r$ and $\mathbf{q_s} \leq \cdots \leq \mathbf{q_2} \leq \mathbf{q_1} (\mathbf{r}, \mathbf{s} \geq \mathbf{1})$.

We extended the zigzag theorem to the class of Gamma semigroups and in this way we give the characterization of Γ-semigroup dominion and throughout this, Γ-semigroups are the member of special class of Γ-semigroups defined by $\mathcal{C}=\{S$ is Γ-semigroup : $a\gamma_1 b = c\gamma_2 d \implies \gamma_1 = \gamma_2$ for all a,b,c,d$\in S$, for all $\gamma_1, \gamma_2 \in \Gamma\}$.

([1, Theorem 3.3]). Let U be a Γ-subsemigroup of a Γ-semigroup S and let $d \in S$. Then $d \in \Gamma - Dom(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:

$$d = a_0\gamma t_1 = y_1\gamma a_1\gamma t_1 = y_1\gamma a_2\gamma t_2 = y_2\gamma a_3\gamma t_2 = \cdots = y_m\gamma a_{2m-1}\gamma t \quad (2)$$

where $m \geq 1$, $a_i \in U$ $(i = 0, 1, \ldots, 2m)$, $y_i, t_i \in S$ $(i = 1, 2, \ldots, m)$, and

$$a_0 = y_1\gamma a_1, \qquad a_{2m-1}\gamma t_m = a_{2m},$$
$$a_{2i-1}\gamma t_i = a_{2i}\gamma t_{i+1}, \qquad y_i\gamma a_{2i} = y_{i+1}\gamma a_{2i+1} \quad (1 \leq i \leq m-1).$$

Such a series of factorization is called a *zigzag* in S over U with value d, length m and spine $a_0, a_1, \ldots, a_{2m}$. We refer to the equations in Result 1.2 as *the zigzag equations*.

Open Problem

1. Full determination of all homotypical semigroup identities whose both sides has repeated variables that are preserved under epis in conjunction with a seminormal identity.
2. Full determination of all heterotypical semigroup identities whose both sides has repeated variables that are preserved under epis in conjunction with a seminormal identity.
3. To prove that some semigroup identities are not preserved under epis in conjunction with a seminormal identity.
4. Extension of Zigzag theorem to different class of semigroups.

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Thank You