# A New Structure Theorem for Locally Inverse Semigroups

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named after the first President of Russia B.N.Yeltsin

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- Further, the morphisms in $\mathbb{L}(S)$ admit a factorisation in the following manner.

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- Nambooripad (1995) extended this to arbitrary regular semigroups using small categories.
- He characterised $\mathbb{L}(S)$ of a regular semigroup $S$ as a normal category, using a set of axioms.

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- The dual $\mathcal{C}^*$ is defined as a full subcategory of the functor category $[\mathcal{C}, \mathbf{Set}]$.

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**Theorem**

*The category $\mathbf{LIS}$ of locally inverse semigroups is equivalent to the category $\mathbf{CUC}$ of cross-connections of unambiguous categories.*

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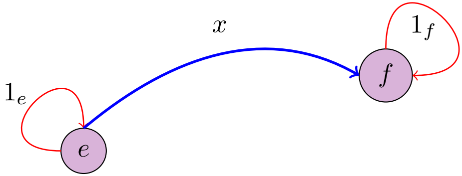

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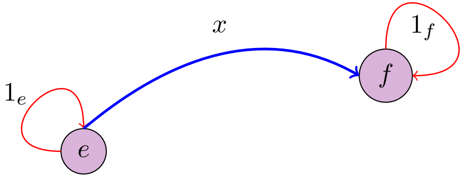


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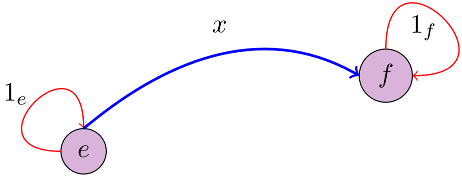


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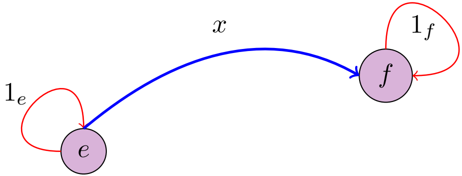


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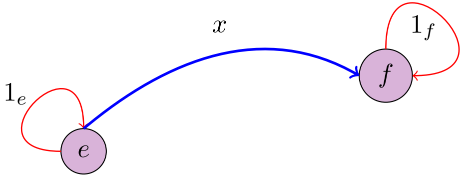


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- Further, the morphisms in groupoid respect the order structure of the vertices. That is, each arrow can be restricted (corestricted) to idempotents below its domain (range).
- Such an associated groupoid is called an inductive groupoid.

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- Identifying a couple more 'local' properties associated with the inverse semigroup, we define define inversive categories as specialisations of unambiguous categories.
- An inversive category is the abstraction of the category $\mathbb{L}(S)$ of the principal left ideals of an inverse semigroup $S$.

## Definition

*A category $\mathcal{C}$ is said to be an inversive category if :*

1. *$\mathcal{C}$ is a so-category ;*
2. *every inclusion in $\mathcal{C}$ splits uniquely ;*
3. *every morphism in $\mathcal{C}$ admits a unique normal factorisation ;*
4. *every morphism in $\langle \mathcal{C} \rangle$ has an inversive factorisation ;*
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**Proposition**

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*The category $\mathbf{IC}$ of inversive categories is equivalent to the category $\mathbf{IG}$ of inductive groupoids.*