Semitopological graph inverse semigroups

Serhii Bardyla

Institute of Mathematics, KGRC, Vienna, Austria

SandGAL, June 13, 2019

Definition

A directed graph $E=(E^0,E^1,r,s)$ consists of sets E^0,E^1 of *vertices* and *edges*, respectively, together with functions $s,r:E^1\to E^0$ which are called *source* and *range*, respectively.

Definition

A bicyclic monoid is the semigroup with the identity 1 generated by two elements p and q subject to the condition $pq=1$.

Definition

For any cardinal λ, the polycyclic monoid P_λ is the semigroup with identity and zero given by the presentation:

$$P_\lambda=\left\langle 0,1,\{p_i\}_{i\in\lambda},\{p_i^{-1}\}_{i\in\lambda}\mid p_ip_i^{-1}=1, p_ip_j^{-1}=0 \text{ for } i\neq j\right\rangle.$$

Remark

Observe that the polycyclic monoid $\mathcal{P}_1$ is isomorphic to the bicyclic monoid with an adjoined zero.

Definition

A directed graph $E = (E^0, E^1, r, s)$ consists of sets E^0, E^1 of *vertices* and *edges*, respectively, together with functions $s, r : E^1 \to E^0$ which are called *source* and *range*, respectively.

Definition

A bicyclic monoid is the semigroup with the identity 1 generated by two elements p and q subject to the condition $pq = 1$.

Definition

For any cardinal λ, the polycyclic monoid P_λ is the semigroup with identity and zero given by the presentation:

$$P_\lambda = \left\langle 0, 1, \{p_i\}_{i\in\lambda}, \{p_i^{-1}\}_{i\in\lambda} \mid p_i p_i^{-1} = 1, p_i p_j^{-1} = 0 \text{ for } i \neq j \right\rangle.$$

Remark

Observe that the polycyclic monoid $\mathcal{P}_1$ is isomorphic to the bicyclic monoid with an adjoined zero.

Definition

A directed graph $E = (E^0, E^1, r, s)$ consists of sets E^0, E^1 of *vertices* and *edges*, respectively, together with functions $s, r : E^1 \to E^0$ which are called *source* and *range*, respectively.

Definition

A bicyclic monoid is the semigroup with the identity 1 generated by two elements p and q subject to the condition $pq = 1$.

Definition

For any cardinal λ, the polycyclic monoid P_λ is the semigroup with identity and zero given by the presentation:

$$P_\lambda = \left\langle 0, 1, \{p_i\}_{i\in\lambda}, \{p_i^{-1}\}_{i\in\lambda} \mid p_i p_i^{-1} = 1, p_i p_j^{-1} = 0 \text{ for } i \neq j \right\rangle.$$

Remark

Observe that the polycyclic monoid $\mathcal{P}_1$ is isomorphic to the bicyclic monoid with an adjoined zero.

Definition

A directed graph $E = (E^0, E^1, r, s)$ consists of sets E^0, E^1 of *vertices* and *edges*, respectively, together with functions $s, r : E^1 \to E^0$ which are called *source* and *range*, respectively.

Definition

A bicyclic monoid is the semigroup with the identity 1 generated by two elements p and q subject to the condition $pq = 1$.

Definition

For any cardinal λ, the polycyclic monoid P_λ is the semigroup with identity and zero given by the presentation:

$$P_\lambda = \left\langle 0, 1, \{p_i\}_{i\in\lambda}, \{p_i^{-1}\}_{i\in\lambda} \mid p_i p_i^{-1} = 1, p_i p_j^{-1} = 0 \text{ for } i \neq j \right\rangle.$$

Remark

Observe that the polycyclic monoid $\mathcal{P}_1$ is isomorphic to the bicyclic monoid with an adjoined zero.

Definition

A graph inverse semigroup $G(E)$ over a directed graph E is a semigroup with zero generated by the sets E^0, E^1 together with a set $E^{-1} = \{e^{-1} | \ e \in E^1\}$ satisfying the following relations for all $a, b \in E^0$ and $e, f \in E^1$:

(i) $a \cdot b = a$ if $a = b$ and $a \cdot b = 0$ if $a \neq b$;

(ii) $s(e) \cdot e = e \cdot r(e) = e$;

(iii) $e^{-1} \cdot s(e) = r(e) \cdot e^{-1} = e^{-1}$;

(iv) $e^{-1} \cdot f = r(e)$ if $e = f$ and $e^{-1} \cdot f = 0$ if $e \neq f$.

Definition

By $\mathrm{Path}(E)$ we denote the set of all pathes of a directed graph E. Observe that each non-zero element of a graph inverse semigroup $G(E)$ is of the form uv^{-1} where $u, v \in \mathrm{Path}(E)$ and $r(u) = r(v)$. A semigroup operation in $G(E)$ is defined in the following way:

$$u_1v_1^{-1} \cdot u_2v_2^{-1} = \begin{cases} u_1wv_2^{-1}, & \text{if } \ u_2 = v_1w \quad \text{for some } w \in \mathrm{Path}(E); \\ u_1(v_2w)^{-1}, & \text{if } \ v_1 = u_2w \quad \text{for some } w \in \mathrm{Path}(E); \\ 0, & \text{otherwise,} \end{cases}$$

and $uv^{-1} \cdot 0 = 0 \cdot uv^{-1} = 0 \cdot 0 = 0$.

Definition

A graph inverse semigroup $G(E)$ over a directed graph E is a semigroup with zero generated by the sets E^0, E^1 together with a set $E^{-1} = \{e^{-1} |\ e \in E^1\}$ satisfying the following relations for all $a, b \in E^0$ and $e, f \in E^1$:

(i) $a \cdot b = a$ if $a = b$ and $a \cdot b = 0$ if $a \neq b$;

(ii) $s(e) \cdot e = e \cdot r(e) = e$;

(iii) $e^{-1} \cdot s(e) = r(e) \cdot e^{-1} = e^{-1}$;

(iv) $e^{-1} \cdot f = r(e)$ if $e = f$ and $e^{-1} \cdot f = 0$ if $e \neq f$.

Definition

By $\mathrm{Path}(E)$ we denote the set of all pathes of a directed graph E. Observe that each non-zero element of a graph inverse semigroup $G(E)$ is of the form uv^{-1} where $u, v \in \mathrm{Path}(E)$ and $r(u) = r(v)$. A semigroup operation in $G(E)$ is defined in the following way:

$$u_1v_1^{-1} \cdot u_2v_2^{-1} = \begin{cases} u_1wv_2^{-1}, & \text{if } \ u_2 = v_1w \quad \text{for some } w \in \mathrm{Path}(E); \\ u_1(v_2w)^{-1}, & \text{if } \ v_1 = u_2w \quad \text{for some } w \in \mathrm{Path}(E); \\ 0, & \text{otherwise}, \end{cases}$$

and $uv^{-1} \cdot 0 = 0 \cdot uv^{-1} = 0 \cdot 0 = 0$.

Remark

For any cardinal λ, polycyclic monoid $\mathcal{P}_\lambda$ is isomorphic to the graph inverse semigroup over a graph E which consists of one vertex and λ distinct loops.

Theorem (Mesyan, Mitchell, 2016)

The following are equivalent for any graph E.

- The only congruences on $G(E)$ are Rees congruences;
- For every $e \in E^1$ there exists $p \in \mathrm{Path}(E) \setminus E^0$ with $s(p) = s(e)$ and $r(p) = r(e)$, such that $p \neq et$ for all $t \in \mathrm{Path}(E)$.

Theorem (Mesyan, Mitchell, 2016)

Each homomorphism h between graphs E_1 and E_2 can be extended to a homomorphism f_h between $G(E_1)$ and $G(E_2)$. Moreover, f_h is surjective iff h is surjective.

Theorem (Mesyan, Mitchell, 2016)

Graph inverse semigroups $G(E_1)$ and $G(E_2)$ are isomorphic iff graphs E_1 and E_2 are isomorphic.

Remark

For any cardinal λ, polycyclic monoid $\mathcal{P}_\lambda$ is isomorphic to the graph inverse semigroup over a graph E which consists of one vertex and λ distinct loops.

Theorem (Mesyan, Mitchell, 2016)

The following are equivalent for any graph E.

- The only congruences on $G(E)$ are Rees congruences;
- For every $e \in E^1$ there exists $p \in \mathrm{Path}(E) \setminus E^0$ with $s(p) = s(e)$ and $r(p) = r(e)$, such that $p \neq et$ for all $t \in \mathrm{Path}(E)$.

Theorem (Mesyan, Mitchell, 2016)

Each homomorphism h between graphs E_1 and E_2 can be extended to a homomorphism f_h between $G(E_1)$ and $G(E_2)$. Moreover, f_h is surjective iff h is surjective.

Theorem (Mesyan, Mitchell, 2016)

Graph inverse semigroups $G(E_1)$ and $G(E_2)$ are isomorphic iff graphs E_1 and E_2 are isomorphic.

Remark

For any cardinal λ, polycyclic monoid $\mathcal{P}_\lambda$ is isomorphic to the graph inverse semigroup over a graph E which consists of one vertex and λ distinct loops.

Theorem (Mesyan, Mitchell, 2016)

The following are equivalent for any graph E.

- The only congruences on $G(E)$ are Rees congruences;
- For every $e \in E^1$ there exists $p \in \mathrm{Path}(E) \setminus E^0$ with $s(p) = s(e)$ and $r(p) = r(e)$, such that $p \neq et$ for all $t \in \mathrm{Path}(E)$.

Theorem (Mesyan, Mitchell, 2016)

Each homomorphism h between graphs E_1 and E_2 can be extended to a homomorphism f_h between $G(E_1)$ and $G(E_2)$. Moreover, f_h is surjective iff h is surjective.

Theorem (Mesyan, Mitchell, 2016)

Graph inverse semigroups $G(E_1)$ and $G(E_2)$ are isomorphic iff graphs E_1 and E_2 are isomorphic.

Remark

For any cardinal λ, polycyclic monoid $\mathcal{P}_\lambda$ is isomorphic to the graph inverse semigroup over a graph E which consists of one vertex and λ distinct loops.

Theorem (Mesyan, Mitchell, 2016)

The following are equivalent for any graph E.

- The only congruences on $G(E)$ are Rees congruences;
- For every $e \in E^1$ there exists $p \in \mathrm{Path}(E) \setminus E^0$ with $s(p) = s(e)$ and $r(p) = r(e)$, such that $p \neq et$ for all $t \in \mathrm{Path}(E)$.

Theorem (Mesyan, Mitchell, 2016)

Each homomorphism h between graphs E_1 and E_2 can be extended to a homomorphism f_h between $G(E_1)$ and $G(E_2)$. Moreover, f_h is surjective iff h is surjective.

Theorem (Mesyan, Mitchell, 2016)

Graph inverse semigroups $G(E_1)$ and $G(E_2)$ are isomorphic iff graphs E_1 and E_2 are isomorphic.

Definition

Let S be a semigroup with zero 0_S and X be a non-empty set. By $\mathcal{B}_X(S)$ we denote the set $X \times S \times X \sqcup \{0\}$ endowed with the following semigroup operation:

$$(a, s, b) \cdot (c, t, d) = \begin{cases} (a, s \cdot t, d), & \text{if } b = c; \\ 0, & \text{if } b \neq c, \end{cases}$$

and $(a, s, b) \cdot 0 = 0 \cdot (a, s, b) = 0 \cdot 0 = 0,$ for each $a, b, c, d \in X$ and $s, t \in S$.

The semigroup $\mathcal{B}_X(S)$ is called the *Brandt $X-$extension of the semigroup S*. Obviously, the set $J = \{(a, 0_S, b) \mid a, b \in X\} \cup \{0\}$ is a two-sided ideal of the semigroup $\mathcal{B}_X(S)$. The Rees factor semigroup $\mathcal{B}_X(S)/J$ is called the *Brandt X^0-extension of the semigroup S* and is denoted by $\mathcal{B}^0_X(S)$. If S is the semilattice $(\{0, 1\}, \min)$ then $\mathcal{B}^0_X(S)$ is called the *semigroup of $X \times X$-matrix units*.

Lemma (Mesyan, Mitchell, 2016)

Each non-zero $\mathcal{D}$-class contains a unique vertex of E.

By D_e we denote the $\mathcal{D}$-class which contains vertex $e \in E^0$. Put $D_e^0 = D_e \cup \{0\}$.

Lemma (B., 2018)

For each vertex e of a graph E, the set D_e^0 is an inverse subsemigroup of $G(E)$.

Lemma (Mesyan, Mitchell, 2016)

Each non-zero $\mathcal{D}$-class contains a unique vertex of E.

By D_e we denote the $\mathcal{D}$-class which contains vertex $e \in E^0$. Put $D_e^0 = D_e \cup \{0\}$.

Lemma (B., 2018)

For each vertex e of a graph E, the set D_e^0 is an inverse subsemigroup of $G(E)$.

Lemma (Mesyan, Mitchell, 2016)

Each non-zero $\mathcal{D}$-class contains a unique vertex of E.

By D_e we denote the $\mathcal{D}$-class which contains vertex $e \in E^0$. Put $D_e^0 = D_e \cup \{0\}$.

Lemma (B., 2018)

For each vertex e of a graph E, the set D_e^0 is an inverse subsemigroup of $G(E)$.

For any vertex e of a graph E put
$I_e = \{u \in \mathrm{Path}(E) \mid r(u) = e\}$;
$Q_e = \{u \in I_e \mid r(v) \neq e$, for each non-trivial prefix v of $u\}$;
$C_e^1 = \{u \in E^1 \mid s(u) = r(u) = e\}$.

Theorem (B., 2018)

Let E be any graph and $e \in E^0$. Then the semigroup D_e^0 is isomorphic to the Brandt Q_e^0-extension of the polycyclic monoid $\mathcal{P}_{|C_e^1|}$.

Corollary (B., 2018)

Let E be a graph which is acyclic at a vertex e. Then the subsemigroup D_e^0 of $G(E)$ is isomorphic to the semigroup of $I_e \times I_e$-matrix units.

For any vertex e of a graph E put
$I_e = \{u \in \mathrm{Path}(E) \mid r(u) = e\}$;
$Q_e = \{u \in I_e \mid r(v) \neq e$, for each non-trivial prefix v of $u\}$;
$C_e^1 = \{u \in E^1 \mid s(u) = r(u) = e\}$.

Theorem (B., 2018)

Let E be any graph and $e \in E^0$. Then the semigroup D_e^0 is isomorphic to the Brandt Q_e^0-extension of the polycyclic monoid $\mathcal{P}_{|C_e^1|}$.

Corollary (B., 2018)

Let E be a graph which is acyclic at a vertex e. Then the subsemigroup D_e^0 of $G(E)$ is isomorphic to the semigroup of $I_e \times I_e$-matrix units.

For any vertex e of a graph E put
$I_e = \{u \in \mathrm{Path}(E) \mid r(u) = e\}$;
$Q_e = \{u \in I_e \mid r(v) \neq e$, for each non-trivial prefix v of $u\}$;
$C_e^1 = \{u \in E^1 \mid s(u) = r(u) = e\}$.

Theorem (B., 2018)

Let E be any graph and $e \in E^0$. Then the semigroup D_e^0 is isomorphic to the Brandt Q_e^0-extension of the polycyclic monoid $\mathcal{P}_{|C_e^1|}$.

Corollary (B., 2018)

Let E be a graph which is acyclic at a vertex e. Then the subsemigroup D_e^0 of $G(E)$ is isomorphic to the semigroup of $I_e \times I_e$-matrix units.

Lemma (Mesyan, Mitchell, 2016)

There exists a one to one correspondence between the set of strongly connected components of a graph E and non-zero $\mathscr{J}$-classes of a GIS $G(E)$. More precisely, $J \cap E^0$ is a strongly connected component of a graph E for each non-zero $\mathscr{J}$-class J of $G(E)$.

By J_A we denote a $\mathscr{J}$-class which contains a strongly connected component $A \subset E^0$.

Lemma (B., 2018)

For any strongly connected component A of a graph E, the set $J_A^0 = J_A \cup \{0\}$ is an inverse subsemigroup of $G(E)$.

Theorem (B., 2018)

Let E be any graph and $A \subseteq E^0$ be a strongly connected component. Then the semigroup J_A^0 is isomorphic to a subsemigroup of the Brandt Q_A^0-extension of the graph inverse semigroup $G(E_A)$ over the induced subgraph E_A.

Lemma (Mesyan, Mitchell, 2016)

There exists a one to one correspondence between the set of strongly connected components of a graph E and non-zero $\mathscr{J}$-classes of a GIS $G(E)$. More precisely, $J \cap E^0$ is a strongly connected component of a graph E for each non-zero $\mathscr{J}$-class J of $G(E)$.

By J_A we denote a $\mathscr{J}$-class which contains a strongly connected component $A \subset E^0$.

Lemma (B., 2018)

For any strongly connected component A of a graph E, the set $J_A^0 = J_A \cup \{0\}$ is an inverse subsemigroup of $G(E)$.

Theorem (B., 2018)

Let E be any graph and $A \subseteq E^0$ be a strongly connected component. Then the semigroup J_A^0 is isomorphic to a subsemigroup of the Brandt Q_A^0-extension of the graph inverse semigroup $G(E_A)$ over the induced subgraph E_A.

Lemma (Mesyan, Mitchell, 2016)

There exists a one to one correspondence between the set of strongly connected components of a graph E and non-zero $\mathcal{J}$-classes of a GIS $G(E)$. More precisely, $J \cap E^0$ is a strongly connected component of a graph E for each non-zero $\mathcal{J}$-class J of $G(E)$.

By J_A we denote a $\mathcal{J}$-class which contains a strongly connected component $A \subset E^0$.

Lemma (B., 2018)

For any strongly connected component A of a graph E, the set $J_A^0 = J_A \cup \{0\}$ is an inverse subsemigroup of $G(E)$.

Theorem (B., 2018)

Let E be any graph and $A \subseteq E^0$ be a strongly connected component. Then the semigroup J_A^0 is isomorphic to a subsemigroup of the Brandt Q_A^0-extension of the graph inverse semigroup $G(E_A)$ over the induced subgraph E_A.

Lemma (Mesyan, Mitchell, 2016)

There exists a one to one correspondence between the set of strongly connected components of a graph E and non-zero $\mathscr{J}$-classes of a GIS $G(E)$. More precisely, $J \cap E^0$ is a strongly connected component of a graph E for each non-zero $\mathscr{J}$-class J of $G(E)$.

By J_A we denote a $\mathscr{J}$-class which contains a strongly connected component $A \subset E^0$.

Lemma (B., 2018)

For any strongly connected component A of a graph E, the set $J_A^0 = J_A \cup \{0\}$ is an inverse subsemigroup of $G(E)$.

Theorem (B., 2018)

Let E be any graph and $A \subseteq E^0$ be a strongly connected component. Then the semigroup J_A^0 is isomorphic to a subsemigroup of the Brandt Q_A^0-extension of the graph inverse semigroup $G(E_A)$ over the induced subgraph E_A.

Theorem (B., 2018)

Each graph inverse semigroup $G(E)$ embeds into a polycyclic monoid $\mathcal{P}_\lambda$, where $\lambda = |G(E)|$ if $|G(E)| > \omega$ and $\lambda = 2$ if $|G(E)| \leq \omega$.

Definition

A *topological* (*inverse*) *semigroup* is a Hausdorff topological space together with a continuous semigroup operation (and an inversion, respectively).

Definition

A *semitopological semigroup* is a Hausdorff topological space together with a separately continuous semigroup operation.

Remark

A semigroup S is semitopological iff for each $x \in S$ left and right shifts $L_x : S \to S$, $L_x(y) = xy$ and $R_x : S \to S$, $R_x(y) = yx$, respectively, are continuous.

Definition

A *topological* (*inverse*) *semigroup* is a Hausdorff topological space together with a continuous semigroup operation (and an inversion, respectively).

Definition

A *semitopological semigroup* is a Hausdorff topological space together with a separately continuous semigroup operation.

Remark

A semigroup S is semitopological iff for each $x \in S$ left and right shifts $L_x : S \to S$, $L_x(y) = xy$ and $R_x : S \to S$, $R_x(y) = yx$, respectively, are continuous.

Definition

A *topological* (*inverse*) *semigroup* is a Hausdorff topological space together with a continuous semigroup operation (and an inversion, respectively).

Definition

A *semitopological semigroup* is a Hausdorff topological space together with a separately continuous semigroup operation.

Remark

A semigroup S is semitopological iff for each $x \in S$ left and right shifts $L_x : S \to S$, $L_x(y) = xy$ and $R_x : S \to S$, $R_x(y) = yx$, respectively, are continuous.

Theorem (Eberhart, Selden, 1969)

A topological bicyclic monoid is discrete.

Theorem (Bertman, West, 1976)

A semitopological bicyclic monoid is discrete.

Theorem (Gutik, 2015)

Locally compact semitopological polycyclic monoid $\mathcal{P}_1$ is either compact or discrete.

Theorem (B.,Gutik, 2016)

Locally compact topological polycyclic monoid $\mathcal{P}_\lambda$ is discrete.

Theorem (Mesyan, Mitchell, Morayne and Péresse, 2016).

Each non-zero element of an arbitrary topological graph inverse semigroup is an isolated point.

Theorem (Eberhart, Selden, 1969)

A topological bicyclic monoid is discrete.

Theorem (Bertman, West, 1976)

A semitopological bicyclic monoid is discrete.

Theorem (Gutik, 2015)

Locally compact semitopological polycyclic monoid $\mathcal{P}_1$ is either compact or discrete.

Theorem (B.,Gutik, 2016)

Locally compact topological polycyclic monoid $\mathcal{P}_\lambda$ is discrete.

Theorem (Mesyan, Mitchell, Morayne and Péresse, 2016).

Each non-zero element of an arbitrary topological graph inverse semigroup is an isolated point.

Previous results

Theorem (Eberhart, Selden, 1969)

A topological bicyclic monoid is discrete.

Theorem (Bertman, West, 1976)

A semitopological bicyclic monoid is discrete.

Theorem (Gutik, 2015)

Locally compact semitopological polycyclic monoid $\mathcal{P}_1$ is either compact or discrete.

Theorem (B.,Gutik, 2016)

Locally compact topological polycyclic monoid $\mathcal{P}_\lambda$ is discrete.

Theorem (Mesyan, Mitchell, Morayne and Péresse, 2016).

Each non-zero element of an arbitrary topological graph inverse semigroup is an isolated point.

Theorem (Eberhart, Selden, 1969)

A topological bicyclic monoid is discrete.

Theorem (Bertman, West, 1976)

A semitopological bicyclic monoid is discrete.

Theorem (Gutik, 2015)

Locally compact semitopological polycyclic monoid $\mathcal{P}_1$ is either compact or discrete.

Theorem (B.,Gutik, 2016)

Locally compact topological polycyclic monoid $\mathcal{P}_\lambda$ is discrete.

Theorem (Mesyan, Mitchell, Morayne and Péresse, 2016).

Each non-zero element of an arbitrary topological graph inverse semigroup is an isolated point.

Theorem (Eberhart, Selden, 1969)

A topological bicyclic monoid is discrete.

Theorem (Bertman, West, 1976)

A semitopological bicyclic monoid is discrete.

Theorem (Gutik, 2015)

Locally compact semitopological polycyclic monoid $\mathcal{P}_1$ is either compact or discrete.

Theorem (B.,Gutik, 2016)

Locally compact topological polycyclic monoid $\mathcal{P}_\lambda$ is discrete.

Theorem (Mesyan, Mitchell, Morayne and Péresse, 2016).

Each non-zero element of an arbitrary topological graph inverse semigroup is an isolated point.

Previous results

Theorem (Mesyan, Mitchell, Morayne and Péresse, 2016)

A locally compact topological graph inverse semigroup over a finite graph is discrete.

Question (Mesyan, Mitchell, Morayne and Péresse, 2016)

Can the above Theorem be generalized to all graphs?

Theorem (Mesyan, Mitchell, Morayne and Péresse, 2016)

A locally compact topological graph inverse semigroup over a finite graph is discrete.

Question (Mesyan, Mitchell, Morayne and Péresse, 2016)

Can the above Theorem be generalized to all graphs?

Theorem (B., 2017)

Discrete topology is the only locally compact semigroup topology on the graph inverse semigroup $G(E)$ if and only if graph E contains a finite amount of vertices and there does not exist a pair of vertices $e, f \in E^0$ such that the set $\{u \in \mathrm{Path}(E) : r(u) = e\}$ is finite and the set $\{a \in E^1 : s(a) = e \text{ and } r(a) = f\}$ is infinite.

Theorem (B., 2018)

Each non-zero element of an arbitrary semitopological graph inverse semigroup is an isolated point.

Theorem (B., 2017)

Discrete topology is the only locally compact semigroup topology on the graph inverse semigroup $G(E)$ if and only if graph E contains a finite amount of vertices and there does not exist a pair of vertices $e, f \in E^0$ such that the set $\{u \in \mathrm{Path}(E) : r(u) = e\}$ is finite and the set $\{a \in E^1 : s(a) = e$ and $r(a) = f\}$ is infinite.

Theorem (B., 2018)

Each non-zero element of an arbitrary semitopological graph inverse semigroup is an isolated point.

Theorem (B., 2018)

Let E be a strongly connected graph which contains a finite amount of vertices. Then a locally compact semitopological graph inverse semigroup $G(E)$ is either compact or discrete.

Corollary (B., 2016)

A locally compact semitopological polycyclic monoid P_λ is either compact or discrete.

Theorem (B., 2018)

Each graph inverse semigroup $G(E)$ admits the coarsest inverse semigroup topology.

Theorem (B., 2018)

Let E be a strongly connected graph which contains a finite amount of vertices. Then a locally compact semitopological graph inverse semigroup $G(E)$ is either compact or discrete.

Corollary (B., 2016)

A locally compact semitopological polycyclic monoid P_λ is either compact or discrete.

Theorem (B., 2018)

Each graph inverse semigroup $G(E)$ admits the coarsest inverse semigroup topology.

Theorem (B., 2018)

Let E be a strongly connected graph which contains a finite amount of vertices. Then a locally compact semitopological graph inverse semigroup $G(E)$ is either compact or discrete.

Corollary (B., 2016)

A locally compact semitopological polycyclic monoid P_λ is either compact or discrete.

Theorem (B., 2018)

Each graph inverse semigroup $G(E)$ admits the coarsest inverse semigroup topology.

Theorem (B., 2018)

By τ_c we denote the topology of Alexandroff one-point compactification of the discrete space $G(E)\setminus\{0\}$ with a narrow 0. For an arbitrary GIS $G(E)$ the following conditions are equivalent:

(1) $(G(E),\tau_c)$ is a topological inverse semigroup;

(2) for each element $uv^{-1}\in G(E)$ the set $M_{uv^{-1}}=\{(ab^{-1},cd^{-1})\in G(E)\times G(E)|\ ab^{-1}\cdot cd^{-1}=uv^{-1}\}$ is finite;

(3) for each vertex e the set $I_e=\{u\in\operatorname{Path}(E)\mid r(u)=e\}$ is finite;

(4) Each $\mathcal{D}$-class in $G(E)$ is finite;

(5) $G(E)$ does not contain isomorphic copies of the bicyclic monoid and an infinite semigroup of matrix units.

Theorem (B., 2018)

For a semitopological GIS $G(E)$ the following conditions are equivalent:

(1) $G(E)$ embeds into a compact topological semigroup S;

(2) $G(E)$ embeds into a sequentially compact topological semigroup S;

(3) $G(E)$ is homeomorphic to $(G(E), \tau_c)$.

Corollary (B., 2018)

Let GIS $G(E)$ be a subsemigroup of a compact topological semigroup. Then the graph E is acyclic and for each vertex e the set $I_e = \{u \in \mathrm{Path}(E) \mid r(u) = e\}$ is finite.

Question

Does the above Theorem holds for countably compact topological semigroups?

Theorem (B., 2018)

For a semitopological GIS $G(E)$ the following conditions are equivalent:

(1) $G(E)$ embeds into a compact topological semigroup S;

(2) $G(E)$ embeds into a sequentially compact topological semigroup S;

(3) $G(E)$ is homeomorphic to $(G(E), \tau_c)$.

Corollary (B., 2018)

Let GIS $G(E)$ be a subsemigroup of a compact topological semigroup. Then the graph E is acyclic and for each vertex e the set
$I_e = \{u \in \mathrm{Path}(E) \mid r(u) = e\}$ is finite.

Question

Does the above Theorem holds for countably compact topological semigroups?

Theorem (B., 2018)

For a semitopological GIS $G(E)$ the following conditions are equivalent:

(1) $G(E)$ embeds into a compact topological semigroup S;

(2) $G(E)$ embeds into a sequentially compact topological semigroup S;

(3) $G(E)$ is homeomorphic to $(G(E), \tau_c)$.

Corollary (B., 2018)

Let GIS $G(E)$ be a subsemigroup of a compact topological semigroup. Then the graph E is acyclic and for each vertex e the set $I_e = \{u \in \mathrm{Path}(E) \mid r(u) = e\}$ is finite.

Question

Does the above Theorem holds for countably compact topological semigroups?

Theorem (Banakh, Dimitrova, Gutik, 2010)

If there is a torsion-free Abelian countably compact topological group G without non-trivial convergent sequences, then there exists a Tychonoff countably compact semigroup S containing a bicyclic semigroup.

Remark

The first example of a group G with properties required in the above Theorem was constructed by M. Tkachenko under the Continuum Hypothesis. Later, the Continuum Hypothesis was weakened to Martin's Axiom by A. Tomita. However, the problem of the existence of a torsion-free countably compact group without convergent sequences in ZFC seems to be open.

Corollary (B., 2018)

Assuming **(MA)**, there exists a countably compact topological semigroup which contains densely the discrete polycyclic monoid $\mathcal{P}_1$.

Theorem (Banakh, Dimitrova, Gutik, 2010)

If there is a torsion-free Abelian countably compact topological group G without non-trivial convergent sequences, then there exists a Tychonoff countably compact semigroup S containing a bicyclic semigroup.

Remark

The first example of a group G with properties required in the above Theorem was constructed by M. Tkachenko under the Continuum Hypothesis. Later, the Continuum Hypothesis was weakened to Martin's Axiom by A. Tomita. However, the problem of the existence of a torsion-free countably compact group without convergent sequences in ZFC seems to be open.

Corollary (B., 2018)

Assuming **(MA)**, there exists a countably compact topological semigroup which contains densely the discrete polycyclic monoid $\mathcal{P}_1$.

Theorem (Banakh, Dimitrova, Gutik, 2010)

If there is a torsion-free Abelian countably compact topological group G without non-trivial convergent sequences, then there exists a Tychonoff countably compact semigroup S containing a bicyclic semigroup.

Remark

The first example of a group G with properties required in the above Theorem was constructed by M. Tkachenko under the Continuum Hypothesis. Later, the Continuum Hypothesis was weakened to Martin's Axiom by A. Tomita. However, the problem of the existence of a torsion-free countably compact group without convergent sequences in ZFC seems to be open.

Corollary (B., 2018)

Assuming **(MA)**, there exists a countably compact topological semigroup which contains densely the discrete polycyclic monoid $\mathcal{P}_1$.

References

1) S. Bardyla: "On a locally compact topological graph inverse semigroups", Topology Appl. (accepted), arXiv:1706.08594.
2) S. Bardyla: "On universal objects in the class of graph inverse semigroups", European Journal of Mathematics, (in press), arXiv:1709.01393v2.
3) S. Bardyla: "On a locally compact semitopological graph inverse semigroups", Mat. Studii, **49** (2018), no.1, p. 19-28.
4) S. Bardyla: "An alternative look at the structure of graph inverse semigroup", Mat. Studii, (in press), arXiv:1806.09671.
5) S. Bardyla: "Embeddings of graph inverse semigroups into compact-like topological semigroups", preprint, arXiv:1810.09169.

Thank You for attention!