An ω-reducibility of $\mathbf{V}$-pointlike pairs for pseudovarieties of ordered monoids $\mathbf{V}$ representing low levels of concatenation hierarchies

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Outline

An ω-reducibility
of $\mathbf{V}$-pointlike pairs
for pseudovarieties of ordered monoids $\mathbf{V}$
representing low levels of concatenation hierarchies

1. **Concatenation hierarchies** of regular languages
2. **Pseudovarieties of ordered monoids** representing levels of concatenation hierarchies
3. **$\mathbf{V}$-pointlike pairs**
4. **ω-reducibility** of $\mathbf{V}$-pointlike pairs
5. New result

Regular Languages

- $A \dots$ finite set (*alphabet*, elements of A are called *letters*)
- $A^* \dots$ a free monoid over A, with the operation of *concatenation*; elements of A^* are called *words*; neutral element ϵ (empty word)
- $L \subseteq A^* \dots$ a *language* over A
- *Regular* language $\dots$ a language created from letters $a \in A$ and ϵ using operations
  - union,
  - *concatenation*,
  - *iteration*.
- The set of all regular languages is closed also under complementation (and hence also under intersection):

$$L \cap K = (L^{\mathrm{C}} \cup K^{\mathrm{C}})^{\mathrm{C}}.$$

Concatenation Hierarchies

$\mathcal{C}_0$ ... a given set of regular languages (over an alphabet A)
$\mathcal{C}_{n+1/2} = Pol(\mathcal{C}_n) = Pol\big(\{L^{\mathrm{C}} \mid L \in \mathcal{C}_{n-1/2}\}\big)$
... *Polynomial closure* = closure under marked concatenation $((K, L) \mapsto K \cdot a \cdot L)$ and union
$\mathcal{C}_{n+1} = B(\mathcal{C}_{n+1/2})$ ... *Boolean* closure (closure under union and complementation)

Example (Straubing–Thérien hierarchy)

- $\mathcal{C}_0 := \{A^*, \emptyset\}$
- $\mathcal{C}_{1/2} = Pol(\mathcal{C}_0)$ ... finite unions of languages of the form

$$A^* a_1 A^* \dots a_n A^* \quad \text{where } a_i \in A$$

- $\mathcal{C}_1 = B(\mathcal{C}_{1/2})$ ... finite Boolean combinations of languages of the form

$$A^* a_1 A^* \dots a_n A^* \quad \text{where } a_i \in A$$

The Main Question

We are given $\mathcal{C}_0$ and $m \in \frac{\mathbb{N}_0}{2}$.

MEMBERSHIP PROBLEM for $\mathcal{C}_m$:

- *Input:* A regular language L.
- *Question:* Does L belong to $\mathcal{C}_m$?

Pseudovarieties of (Ordered) Monoids

- For every finite alphabet A, have a set of regular languages $\mathcal{C}_0(A)$.
- If the sets $\mathcal{C}_0(A)$ satisfy some nice properties, they form a *variety* of regular languages.
  Then
  - full levels of conc. hierarchy $\mathcal{C}_n$ = varieties of regular languages $\leftrightarrow$ *pseudovarieties* V_n of finite monoids,
  - half levels of conc. hierarchy $\mathcal{C}_{n+1/2}$ = *positive* varieties of regular languages $\leftrightarrow$ pseudovarieties $\mathrm{V}_{n+1/2}$ of finite *ordered* monoids.

Example (Straubing–Thérien hierarchy)

- $\mathcal{C}_0(A) = \{A^*, \emptyset\}$ ... $\mathrm{V}_0 =$ trivial pseudovariety
- $\mathcal{C}_{1/2}(A) = Pol(\mathcal{C}_0(A))$ ... $\mathrm{V}_{1/2} = [[1 \leq a]]$
- $\mathcal{C}_1(A) = B(\mathcal{C}_{1/2}(A))$ ... $\mathrm{V}_1 =$ pseudovariety of $\mathcal{J}$-trivial monoids (Simon, 1975)

Membership Problem for Pseudovarieties

The membership problem for levels of concatenation hierarchies

- How to decide whether a given regular language belongs to $\mathcal{C}_m$ for given $\mathcal{C}_0$ and $m \in \frac{\mathbb{N}_0}{2}$?

…corresponds to the membership problem for pseudovarieties of finite (ordered) monoids:

- How to decide whether a given finite (ordered) monoid belongs to V_m for given V_0 and $m \in \frac{\mathbb{N}_0}{2}$?

Half Levels

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "Pol"] & \mathcal{C}_{1/2} \arrow[d] & \mathcal{C}_{3/2} \arrow[d] & \mathcal{C}_{5/2} \\
& \overline{\mathcal{C}_{1/2}} \arrow[ur, "Pol"] & \overline{\mathcal{C}_{3/2}} \arrow[ur, "Pol"] & \cdots
\end{tikzcd}$$

where $\overline{\mathcal{C}} = \{A^* \setminus L \mid L \in \mathcal{C}\}$.

Pointlike Pairs for Half Levels

Theorem (Branco, Pin, 2009, Place, Zeitoun, 2014, 2017, Almeida, Bartoňová, Klíma, Kunc, 2015)

$$\mathrm{V}_{n+1/2} = [[x^{\omega+1} \leq x^\omega y x^\omega \mid \mathrm{V}_{n-1/2} \models y \leq x]]$$

- $x, y \in \overline{\Omega}_A \mathrm{M}$, where $\overline{\Omega}_A \mathrm{M}$ is a free profinite monoid over some finite alphabet A,
- $\mathrm{V}_{n-1/2} \models y \leq x \;\Leftrightarrow\;$ for every finite ordered monoid $M \in \mathrm{V}_{n-1/2}$ and for every continuous homomorphism $\varphi\colon \overline{\Omega}_A \mathrm{M} \to M$, we have $\varphi(y) \leq \varphi(x)$.

So to decide whether $M \in \mathrm{V}_{n+1/2}$, it suffices to compute the set

$$\mathbf{V}_{n-\frac{1}{2}}[M] := \big\{(s,t) \in M \times M \mid \exists x, y \in \overline{\Omega}_A \mathrm{M},$$

$$\exists \text{ cont. hom. } \varphi\colon \overline{\Omega}_A \mathrm{M} \to M\colon s = \varphi(y), t = \varphi(x), \mathrm{V}_{n-\frac{1}{2}} \models y \leq x\big\}.$$

...the set of all "$\mathrm{V}_{n-\frac{1}{2}}$-*pointlike pairs*" (an ordered analogue of V-*pointlike pairs*, where V is a pseudovariety of ordinary monoids).

ω-reducibility

$$\mathbf{V}_{\boldsymbol{n}-\frac{1}{2}}[\boldsymbol{M}] := \big\{(s,t) \in M \times M \mid \exists x, y \in \overline{\Omega}_A\mathrm{M},$$
$$\exists \text{ cont. hom. } \varphi\colon \overline{\Omega}_A\mathrm{M} \to M\colon s = \varphi(y), t = \varphi(x), \mathrm{V}_{n-\frac{1}{2}} \models y \leq x\big\}.$$

By Almeida, Steinberg (2000), it suffices to prove

1. the decidability of the ω-inequality (= inequality of ω-words) problem for $\mathrm{V}_{n-\frac{1}{2}}$ (done by Almeida, Klíma, Kunc, 2018, for conc. hierarchies of *star-free* languages),
2. an *ω-reducibility* of $\mathrm{V}_{n-\frac{1}{2}}[M]$, i.e. that the following property holds:

$$\mathrm{V}_{n-\frac{1}{2}}[M] = \big\{(s,t) \in M \times M \mid \exists x, y \in \boldsymbol{\Omega}^{\boldsymbol{\omega}}_{\boldsymbol{A}}\mathbf{M},$$
$$\exists \text{ cont. hom. } \varphi\colon \Omega^{\omega}_A\mathrm{M} \to M\colon s = \varphi(y), t = \varphi(x), \mathrm{V}_{n-\frac{1}{2}} \models y \leq x\big\}.$$

Known and New Results

- The sets of V_m-pointlike pairs are known to be algorithmically computable for $m = 0, \frac{1}{2}, 1, \frac{3}{2}$, where V_0 is a locally finite pseudovariety. (Place, Zeitoun, 2017 – proof as elementary as possible – done in locally finite pseudovarieties forming a stratification of pseudovariety V_m.)
- By use of ideas from a proof of corectness of an algorithm computing $\mathrm{V}_{\frac{1}{2}}[M]$ by Place, Zeitoun we have proved the ω-reducibility of this set. (using some "profinite tools")

Theorem (J.B.)

Let V_0 be a locally finite pseudovariety, M a finite monoid. Then the set $\mathrm{V}_{\frac{1}{2}}[M]$ of $\mathrm{V}_{1/2}$-pointlike pairs of M is ω-reducible.

Corollary

Theorem (J.B.)

Let V_0 be a locally finite pseudovariety, M a finite monoid. Then the set $\mathrm{V}_{\frac{1}{2}}[M]$ of $\mathrm{V}_{1/2}$-pointlike pairs of M is ω-reducible.

Theorem (Branco, Pin, 2009, Place, Zeitoun, 2014, 2017, . . .)

$$\mathrm{V}_{n+1/2} = [[x^{\omega+1} \leq x^\omega y x^\omega \mid \mathrm{V}_{n-1/2} \models y \leq x]]$$

Corollary (J.B.)

Let V_0 be a locally finite pseudovariety. Then

$$\mathrm{V}_{3/2} = [[x^{\omega+1} \leq x^\omega y x^\omega \mid \exists \text{ finite } A\colon x, y \in \Omega_A^\omega \mathrm{M}, \mathrm{V}_{1/2} \models y \leq x]].$$

In particular, the pseudovariety $\mathrm{V}_{3/2}$ has a basis of inequalities of ω-words.

Case of Straubing–Thérien Hierarchy

$\mathrm{W}_0 \ldots$ trivial pseudovariety
It is known that $\mathrm{W}_{3/2} = \mathrm{V}_{1/2}$, where $\mathrm{V}_0 = \mathrm{Sl}$ (the pseudovariety of semilattices), which is still locally finite.

Corollary (J.B.)

$\mathrm{W}_{\frac{5}{2}} = [[x^{\omega+1} \leq x^\omega y x^\omega \mid \exists \textit{ finite } A\colon x, y \in \Omega_A^\omega \mathrm{M}, \mathrm{W}_{\frac{3}{2}} \models y \leq x]].$

In particular, the pseudovariety $\mathrm{W}_{5/2}$ of ST-hierarchy has a basis of inequalities of ω-words.

Further, using a characterization of W_2 of ST-hierararchy by a set of pseudoidentities parametrized by a generalization of $\mathrm{W}_{3/2}$-pointlike pairs (Place, Zeitoun, 2017), we obtain the following.

Corollary (J.B.)

The pseudovariety W_2 of ST-hierarchy has a basis of pseudoidentities of ω-words.

Current and Future Work

- Generalization of our profinite approach to obtain easier/ shorter proofs and to extend it to higher levels of concatenation hierarchies.

Thank you for your attention.