Amalgams and HNN Extensions Of Inverse Semigroups

Paul Bennett, Singapore

June 6, 2019

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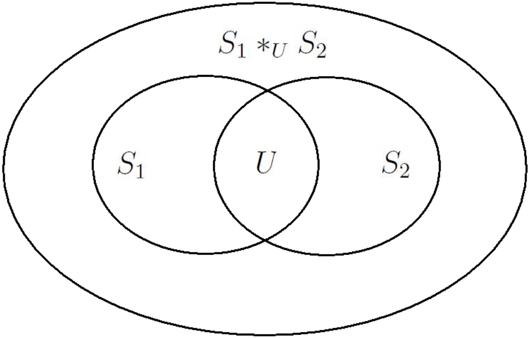

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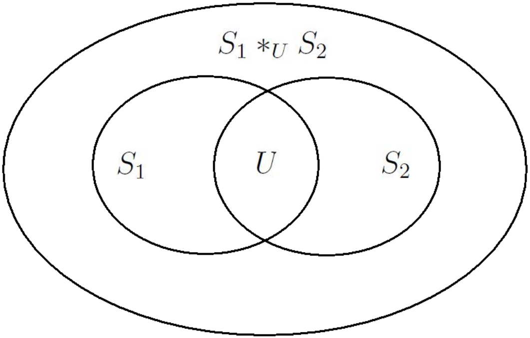
- Hall (1975) showed that S_1 and S_2 are embedded into the amalgamated free product $S_1 *_U S_2$, with their images intersecting in U.



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Lower Bounded Subsemigroups

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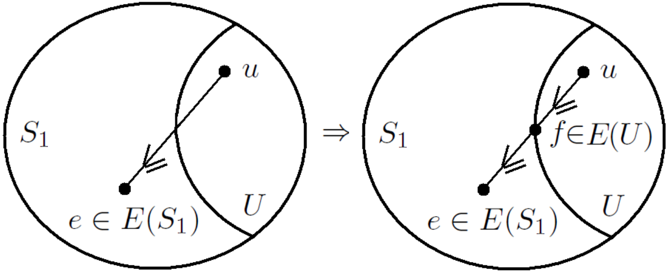

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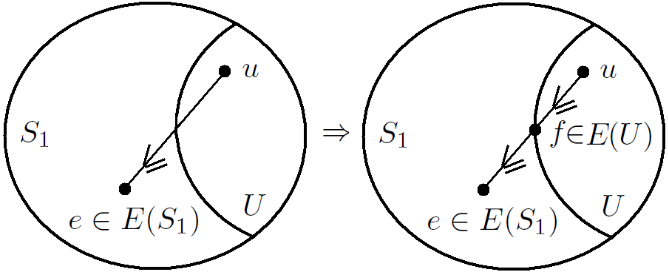

- Similarly, we say that U is *lower bounded* in S_2 if $u \geq e$, where $u \in U$ and $e \in E(S_2)$, implies $u \geq f \geq e$, for some $f \in E(U)$.

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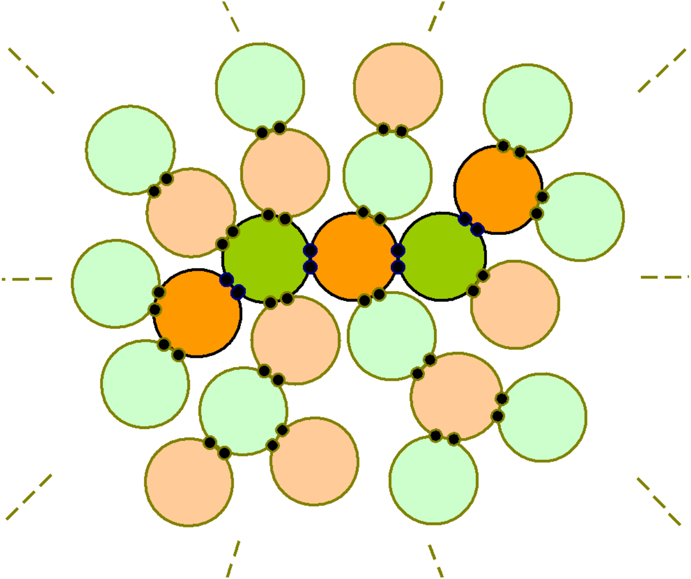
The results generalise Bennett's papers on lower bounded amalgams (1997).

The Schützenberger Graphs Of $S_1 *_U S_2$



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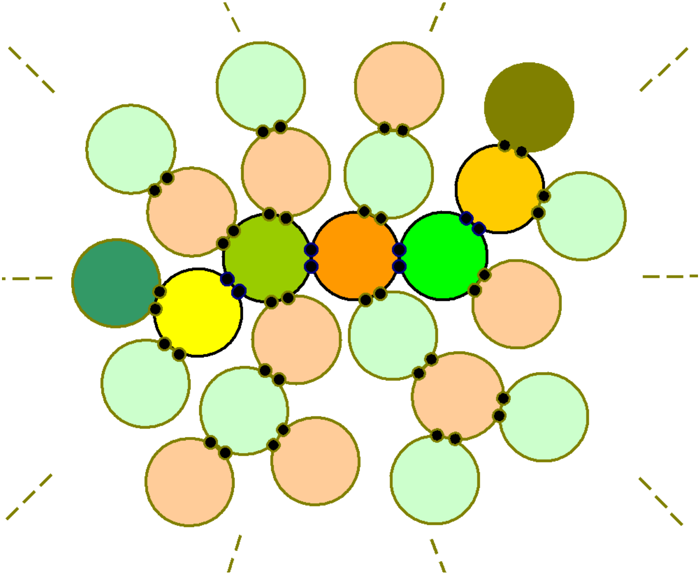

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- All definitions of μ_U agree on $U *_{E(U)} M(U)$.

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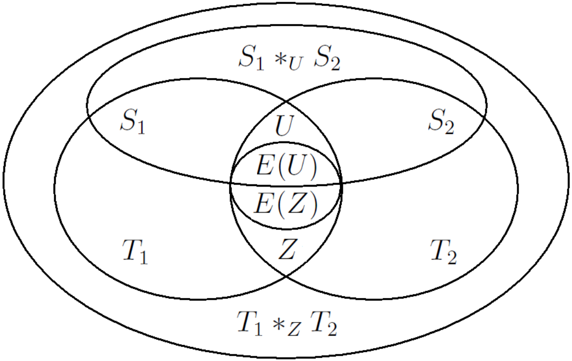

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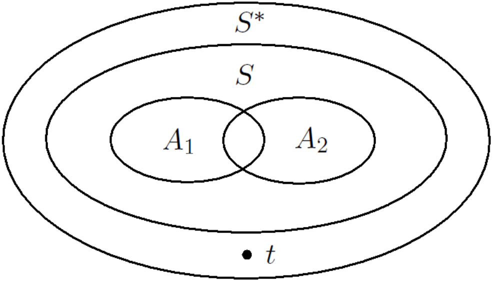
HNN Extensions

- Let S be an inverse semigroup and let $\phi : A_1 \rightarrow A_2$ be an isomorphism between inverse subsemigroups of S.



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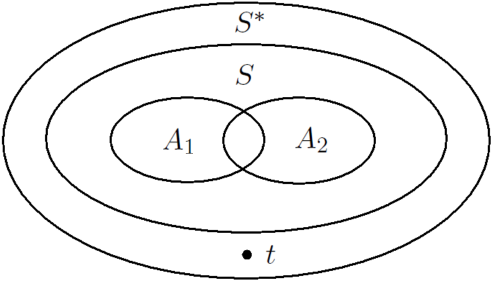

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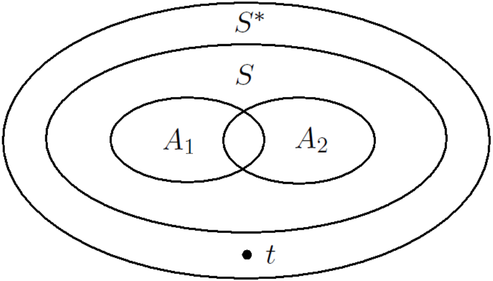
- If A_1 and A_2 are not monoids then we adjoin a shared identity 1 to A_1, A_2 and S. Let e_i be the identity of A_i, for $i = 1, 2$.
- Yamamura (1997) showed that the HNN extension $S^* = [S; A_1, A_2; \phi]$ contains a copy of S and an element t, with $tt^{-1} = e_1$, $t^{-1}t = e_2$ and $t^{-1}at = (a)\phi$, for all $a \in A_1$.



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The results generalise Jaycayová's PhD on lower bounded HNN extensions (1997).

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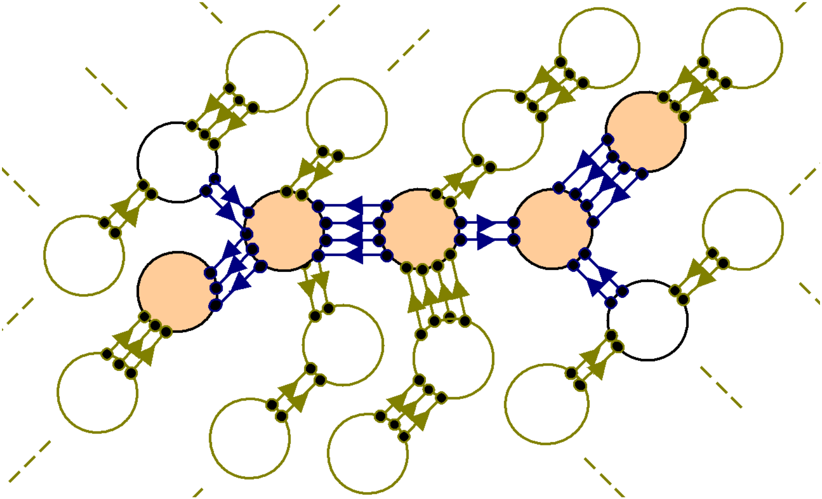

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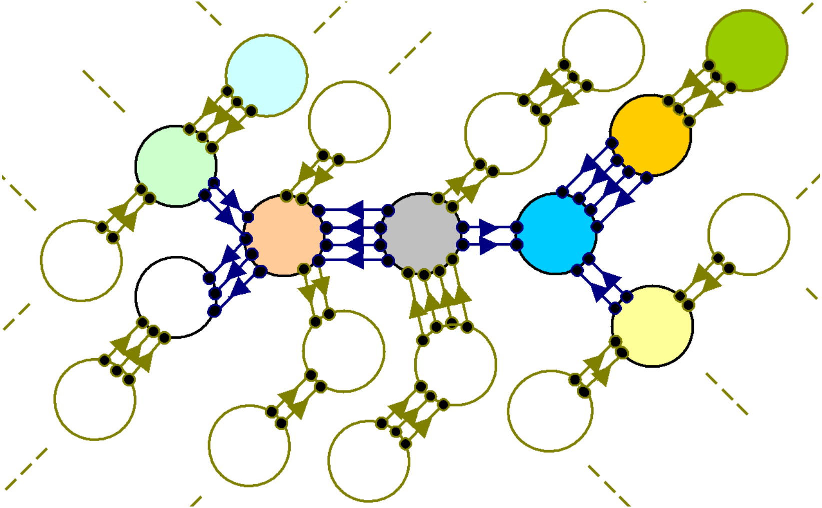

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- We have an isomorphism $\pi : Z_1 \to Z_2$ and S is embedded into T_1 and T_2. Put $T = T_1 *_S T_2$.

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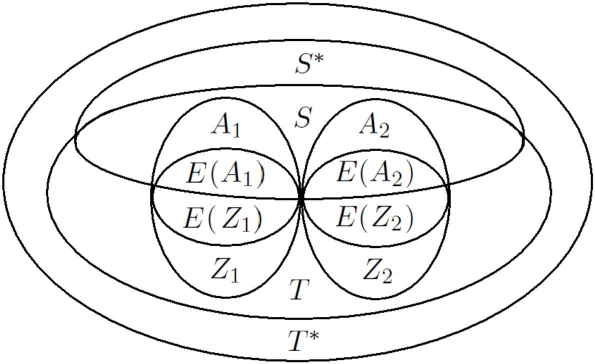

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- Put $S_1 = S *_{\{e_1\}} FIM(x_1)$, where $\{e_1\} \cong \{x_1 x_1^{-1}\}$. Let U_1 be the inverse subsemigroup of S_1 generated by $S \cup x_1^{-1} A_1 x_1$.

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- Put $S_1 = S *_{\{e_1\}} FIM(x_1)$, where $\{e_1\} \cong \{x_1 x_1^{-1}\}$. Let U_1 be the inverse subsemigroup of S_1 generated by $S \cup x_1^{-1} A_1 x_1$.
- Put $S_2 = S *_{\{e_2\}} FIM(x_2)$, where $\{e_2\} \cong \{x_2^{-1} x_2\}$. Let U_2 be the inverse subsemigroup of S_2 generated by $S \cup x_2 A_2 x_2^{-1}$.

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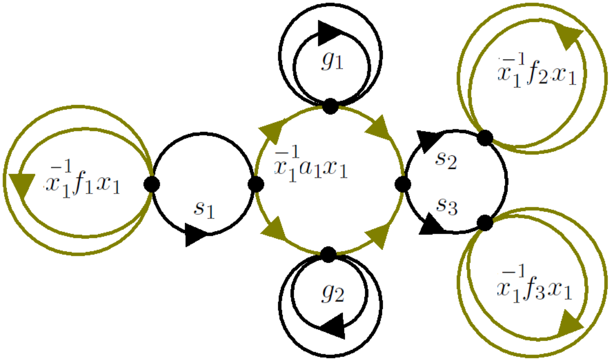
HNN Theorem

The HNN extension S^* is embedded onto the inverse subsemigroup of $S_1 *_{U_1 \cong U_2} S_2$ generated by S and $t = x_1 x_2$.

Sketch Of Proof, Part 1 / 4

A typical Schützenberger graph of $S * x_1^{-1} A_1 x_1$:

- Black circles represent Schützenberger graphs of S.



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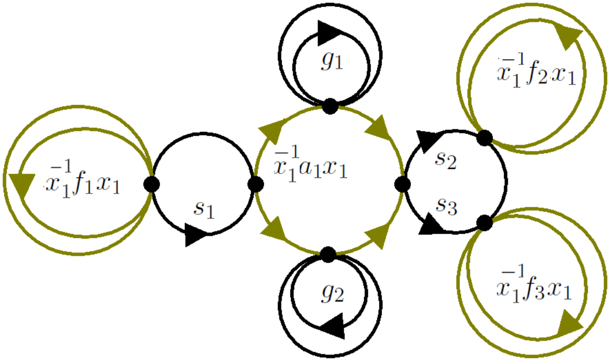

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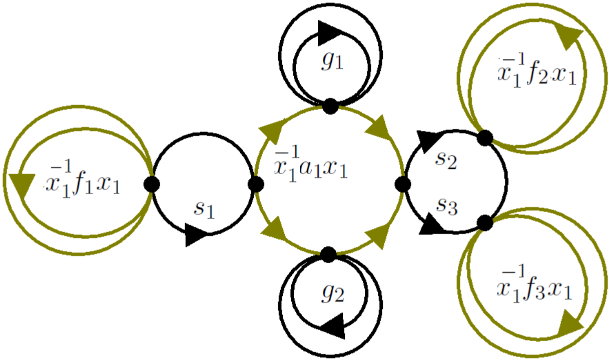

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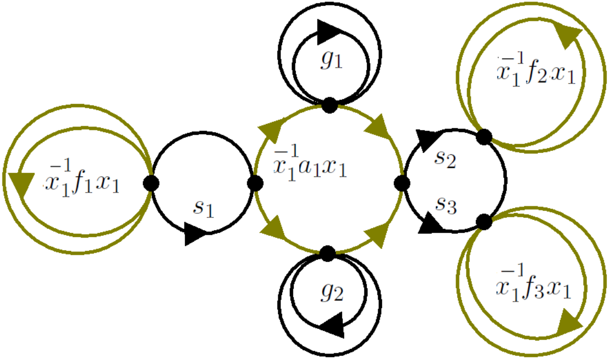
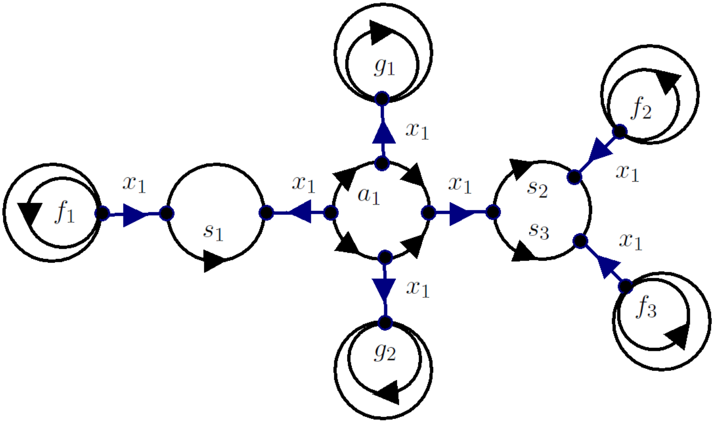
- $s_1, s_2, s_3 \in S$, $g_1, g_2 \in E(S)$, $a_1 \in A_1$, $f_1, f_2, f_3 \in E(A_1)$.



Sketch Of Proof, Part 2 / 4

- Replace the green circles by the corresponding Schützenberger graphs of $S * FIM(x_1)$.



Sketch Of Proof, Part 3 / 4

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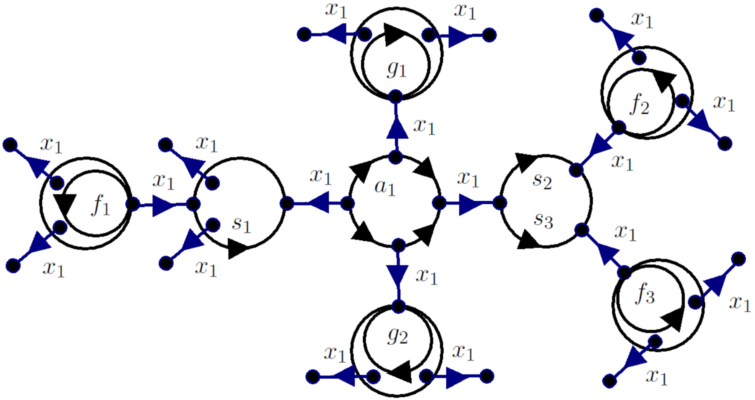

Sketch Of Proof, Part 3 / 4

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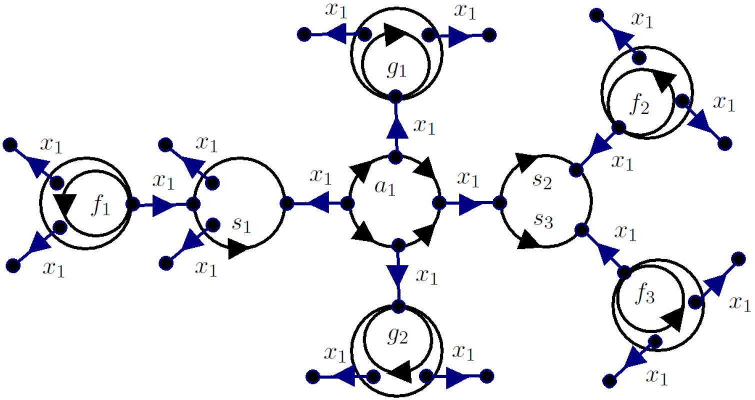

Sketch Of Proof, Part 4 / 4

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- This completes the proof.

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- We can prove analogues of group theory results, such as the HNN embedding theorem.