Groups with vanishing class size p

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Cremona, June 2019

Some general notation

In this talk, every group is assumed to be a finite group. If the character table of G is known, then some deep structural information on the group G itself can be deduced and many results in the literature show that the distribution of zeros in the character table is significant in this context.

Historical overview

- The study of conjugacy class sizes in finite groups goes back to the beginning of character theory, when Burnside proved that a simple group does not have a conjugacy class size that is a prime power.
- Ito began the study of the structure of a group in terms of the number of conjugacy class sizes in (1953) in his paper "On finite groups with given conjugacy types,I", where he proved that a group having only two conjugacy class sizes must be the direct product of a p-group and an abelian group.

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- Ishikawa had a major breakthrough in his paper "On finite p-groups which have only two conjugacy lenghts", where he proved that a p-group with only two conjugacy class sizes has nilpotency class at most 3 and furthermore he shows in his paper "Finite p-groups up to isoclonism,which have only two conjugacy lengths" that a p-group has conjugacy class sizes $1, p$ if and only if the group is isoclinic to an extra-special group.
- The study of group elements that vanish on an irreducible character also goes back to 1903 when Burnside showed that

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Notation

With this in mind we say that an element $g \in G$ is a vanishing element if there exists an irreducible character χ such that $\chi(g) = 0$ and we denote by $vcs(G)$ the set of conjugacy class sizes of the so-called vanishing elements of G. This is of course a set of positive integers related to the finite group G. Recently there has been more concentrated study on the vanishing elements and classes in a group. This began with the work of Isaacs, Navarro and Wolf "Finite group elements where no irreducible character vanishes" where they proved

Theorem

If G is a finite soluble group and P is a Sylow 2-subgroup of G, $F(G)$ the Fitting subgroup of G then every element outside of $PF(G)$ is a vanishing element.Besides if G is nilpotent every element outside $Z(G)$ is vanishing..

Now for the coming developments it is necessary to remind some notation of graph theory.

Let $\mathcal{G}$ be a graph with $V(\mathcal{G})$ set of vertices and $E(\mathcal{G})$ set of edges. We will denote with $n(\mathcal{G})$ the number of connected components of $\mathcal{G}$ and if x and y are vertices in the same component we define a *distance* $\mathrm{d}(x, y)$ between x and y being the number of edges in a path between x and y with the fewer number of edges. The *diameter* of $\mathcal{G}$ is the maximum distance between them.

In fact to carry out a deeper analysis of the "arithmetical structure" of $vcs(G)$, we attach a particular graph to this set.

Let X be a finite nonempty subset of $\mathbb{N}$:

- The *prime graph* $\Delta(X)$ is defined as follows:

$$\mathrm{Vert}(\Delta(X)) = \{p \text{ prime } \ : \ \exists x \in X \text{ with } p|x\},$$

  and two vertices p, q are adjacent if there exists $x \in X$ with $pq|x$. Now let us suppose $X{=}vcs(G)$ and study

$\Delta(vcs(G))$

Dolfi ans Sanus proved the following

Theorem

If a prime number p is not a vertex of $\Delta(vcs(G))$ then G has a normal p-complement and abelian Sylow p-subgroups. We also observe that the vertex set of $\Delta(vcs(G))$ can be smaller than that of $\Delta(cs(G))$. Yet if one assume that G has a non abelian minimal normal subgroup, then the two vertex sets coincide and in this situation the absence of an edge in the graph $\Delta(vcs(G))$ reflects in the normal structure of G.

The study of vanishing conjugacy class sizes in nonsoluble groups was developed in the following

Theorem

(BBCP) Let G be a finite group and suppose that G has a non abelian minimal normal subgroup. If p and q are vertices of $\Delta(vcs(G))$ but there is no vanishing conjugacy class of G whose size is divisible by pq, then G is $\{p, q\}$-soluble.

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$\Delta(vcs(G))$

A consequence of the previous theorem is that (under the same assumptions) if a vertex p of $\Delta(vcs(G))$ is not complete, then the group G is p-soluble.

Moreover, if the group G has no abelian normal subgroups, then the graph $\Delta(vcs(G))$ is complete.

Theorem

(BBCP) Let G be a finite group with trivial Fitting subgroup.Then every prime divisors of $|G|$ is a vertex of $\Delta(vcs(G))$, and $\Delta(vcs(G))$ is a complete graph

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- Following the lead of Ishikawa together with Lewis and Pacifici we classify those groups where the only vanishing class size is p, obtaining the following result:

Theorem

Let G be a group and let p be a prime. The vanishing classes of G all have size p if and only if one of the following occurs:

1. $G = P \times H$ *where P is a p-group with* $\mathrm{cs}(P) = \{1, p\}$ *and H is an abelian p'-group.*
2. $G/Z(G)$ *is a Frobenius group whose Frobenius kernel has order p.*