Congruences on $G \wr \mathcal{I}_n$

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SandGAL June 2019

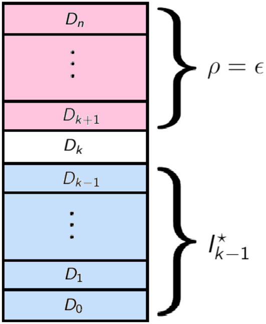
Introduction

- Aim: To understand congruences on semigroups that "look like transformation monoids"
- One direction involves looking at diagram monoids
  - *Congruence lattices of finite diagram monoids; 2018; East, Mitchell, Ruškuc, Torpey*
- $\mathcal{I}_n$ is the partial automorphism of an independence algebra
- A free group act is an independence algebra, is it possible to describe congruences on its partial automorphism monoid?

Congruences on $\mathcal{I}_n$

For each $a, b \in D_k$ with $a \,\mathcal{H}\, b$, there is $\mu \in \mathcal{S}_k$ such that a, b have the following form:

$$a = \begin{pmatrix} b_1 & b_2 & \dots & b_k \\ a_1 & a_2 & \dots & a_k \end{pmatrix}, \quad b = \begin{pmatrix} b_1 & b_2 & \dots & b_k \\ a_{1\mu} & a_{2\mu} & \dots & a_{k\mu} \end{pmatrix}$$



For $k \leq n$ and $N \trianglelefteq \mathcal{S}_k$ define ρ_N as follows:

- $a \;\rho_N\; a$ for all a;
- $a \;\rho_N\; b$ for a, b with $\mathrm{rk}(a), \mathrm{rk}(b) < k$;
- for $\mathrm{rk}(a) = k = \mathrm{rk}(b)$, $a \;\rho_N\; b$ if $a \,\mathcal{H}\, b$ and $\mu \in N$.

$G \wr \mathcal{I}_n$

Definition

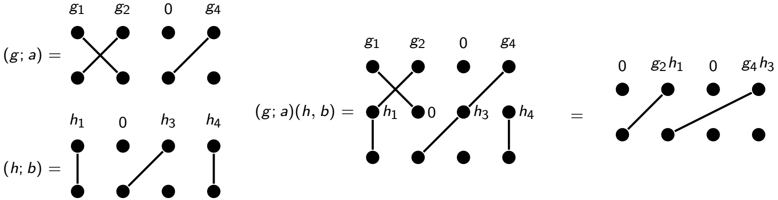
We define the partial wreath product of G with $\mathcal{I}_n$ as follows

$$G \wr \mathcal{I}_n = \{(g; a) \in (G^0)^n \times \mathcal{I}_n \mid g_i \neq 0 \iff i \in \mathrm{Dom}(a)\}.$$

Multiplication is defined as

$$(g_1, \ldots, g_n; a)(h_1, \ldots, h_n; b) = (g_1 h_{1a}, \ldots, g_n h_{na}; ab),$$

letting $g0 = 0 = 0g$ for all $g \in G$, and $h_{ia} = 0$ when ia is undefined.



The Basic Structure of $G \wr \mathcal{I}_n$

$$E(G \wr \mathcal{I}_n) = \{(1^e; e) \mid e \in E(\mathcal{I}_n)\} \cong E(\mathcal{I}_n)$$

where $1^e \in (G \cup \{0\})^n$ has: $1^e_i = 1$ if $i \in \mathrm{Dom}(e)$, and $1^e_i = 0$ otherwise.

Green's relations are induced by those for $\mathcal{I}_n$

$$(g; a) \ \mathcal{K}^{G \wr \mathcal{I}_n} \ (h; b) \iff a \ \mathcal{K}^{\mathcal{I}_n} \ b.$$

Ideals of $G \wr \mathcal{I}_n$ are

$$I_k = \{(g; a) \in G \wr \mathcal{I}_n \mid \mathrm{rk}(a) \leq k\}$$

for each $0 \leq k \leq n$, where $\mathrm{rk}(a) = |\mathrm{Dom}(a)|$.
Write $I_k^\star$ for the corresponding Rees congruence on $G \wr \mathcal{I}_n$.

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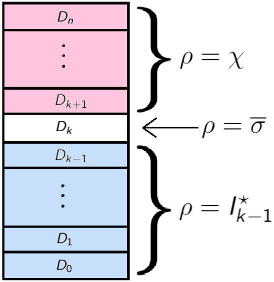
Congruence decomposition



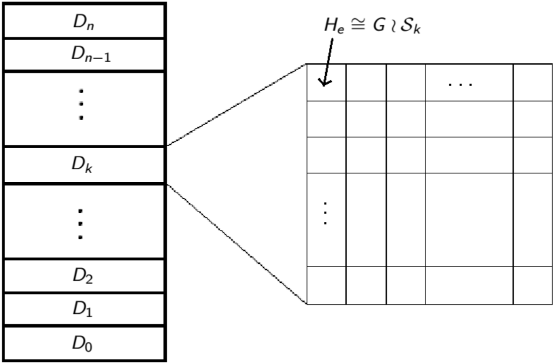
Theorem (Lima, 1993)

Let ρ be a congruence on $G \wr \mathcal{I}_n$. If ρ is not the universal congruence on $G \wr \mathcal{I}_n$ then there are $k \leq n$, σ a non universal relation on I_k/I_{k-1} and χ an idempotent separating congruence such that

$$\rho = I^\star_{k-1} \cup \overline{\sigma} \cup \chi.$$

Where for σ a congruence on I_k/I_{k-1} let

$$\overline{\sigma} = \{(a, b) \in D_k \times D_k \mid (a/I_{k-1}, b/I_{k-1}) \in \sigma\}$$



The lattice of non universal congruences on I_k/I_{k-1} is isomorphic to the lattice of normal subgroups of $G \wr \mathcal{S}_k$.

Normal subgroups of $G \wr \mathcal{S}_k$

Definition

$J \leq G^k$ is (permutation) invariant if for all $\sigma \in \mathcal{S}_k$ we have that

$$(g_1, g_2, \ldots, g_k) \in J \iff (g_{1\sigma}, g_{2\sigma}, \ldots, g_{k\sigma}) \in J$$

If $Z \trianglelefteq G \wr \mathcal{S}_k$, then $J(Z)$ is an invariant normal subgroup of G^k, where

$$J(Z) = \{j \in G^k \mid (j, 1) \in Z\}$$

Theorem (Usenko, 1991)

The normal subgroups of $G \wr \mathcal{S}_k$ are exactly:

(1) $\{(j, 1) \mid j \in J\}$ *for* $J \trianglelefteq G^k$ *an invariant subgroup;*

(2) $\{(x, q) \mid q \in Q, xJ = q\theta\}$ *where* $Q \trianglelefteq \mathcal{S}_k$, $J \trianglelefteq G^k$ *is an invariant subgroup such that the induced action of* $\mathcal{S}_k$ *on* G^k/J *is trivial, and* $\theta : Q \to G^k/J$ *is a homomorphism.*

Invariant Subgroups of G^k

Definition

Let G be a group, $N \trianglelefteq M \trianglelefteq K$ be normal subgroups of G, and $\theta : K/N \to M/N$ an homomorphism. Then $\{K, M, N, \theta\}$ is a k-invariant quadruple for G if

(i) $[G, K] \subseteq N$;

(ii) $\mathrm{Im}(\theta) \subseteq M \subseteq \{x \in K/N \mid x\theta = x^{-k}\}$.

Given an k-invariant quadruple we define the following subset of G^k :

$$\mathbf{J}_k(K, M, N, \theta) = \{(g_1, \ldots, g_k) \in K^k \mid g_1 M = \cdots = g_k M,$$
$$g_1 N\theta = g_1^{1-k} g_2 g_3 \ldots g_k N\}.$$

Theorem

These are exactly the invariant normal subgroups of G^k.

Invariant Subgroups of G^k

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Corollary

If G is a finite group then there is an integer $\lambda(G)$ such that for each k the number of permutation invariant subgroups of G^k is less than $\lambda(G)$.

Normal subgroups of $G \wr \mathcal{S}_k$

Theorem (Usenko, 1991)

The normal subgroups of $G \wr \mathcal{S}_k$ are exactly:

(1) $\{(j, 1) \mid j \in J\}$ *for* $J \trianglelefteq G^k$ *an invariant subgroup;*

(2) $\{(x, q) \mid q \in Q, xJ = q\theta\}$ *where* $Q \trianglelefteq \mathcal{S}_k$, $J = \mathbf{J}_k(G, G, N, xN \mapsto x^{-n}N)$, *and* $\theta : Q \to G^k/J$ *is a homomorphism.*

Corollary

Let G be a finite group, then there is finite number $\lambda_2(G)$ such that for all n the number of normal subgroups of $G \wr \mathcal{S}_k$ is at most $\lambda_2(G)$.

Idempotent separating congruences

The centraliser of E is the set $E\zeta = \{a \mid \forall e \in E \; ae = ea\}$.
For $G \wr \mathcal{I}_n$

$$E\zeta = \{(g; e) \in G \wr \mathcal{I}_n \mid e \in E(\mathcal{I}_n)\}.$$

Theorem (Petrich, 1978)

Let S be an inverse semigroup. The lattice of idempotent separating congruences on S is isomorphic to the lattice of full, self-conjugate, inverse subsemigroups of S contained in $E\zeta$.
The following maps are mutually inverse lattice isomorphisms:

$$K \mapsto \rho = \{(a, b) \mid a^{-1}a = b^{-1}b, \; ab^{-1} \in K\},$$
$$\rho \mapsto \ker(\rho) = \{a \in S \mid \exists e \in E(S) \; with \; e \; \rho \; a\}$$

Proposition

For $1 \leq i \leq n$ let $J_i \trianglelefteq G^i$ be invariant normal subgroups such that

$$\{(g_1, \ldots, g_{k-1}) \mid \exists g_k \in G \text{ with } (g_1, \ldots, g_{k-1}, g_k) \in J_k\} \subseteq J_{k-1}.$$

Let $\pi : (G^0)^n :\to \bigsqcup_{0 \leq i \leq n} G^i$ be the function that ignores all 0 entries. Let

$$K = \bigcup_{e \in E(\mathcal{I}_n)} \{(g_1, \ldots, g_n; e) \in G \wr \mathcal{I}_n \mid g\pi \in J_i \text{ where } i = \mathrm{rk}(e)\}.$$

Then K is a full, self conjugate inverse subsemigroup of $G \wr \mathcal{I}_n$ with $K \subseteq E\zeta$.
Moreover, every such subsemigroup arises in this way.

$$\{(g_1, \ldots, g_{k-1} \mid \exists g_k \text{ with } (g_1, \ldots, g_k) \in \mathbf{J}_k(K, M, N, \theta)\}$$
$$= \mathbf{J}_{k-1}(K, M, M, xM \mapsto M)$$

$$
\begin{tikzcd}
K_1 & K_2 \arrow[l] & K_3 \arrow[l] & \cdots \arrow[l, dotted] & K_{n-2} \arrow[l, dotted] & K_{n-1} \arrow[l] & K_n \arrow[l] \\
 & & M_3 \arrow[u] \arrow[dl] & \cdots \arrow[u, dotted] \arrow[dl, dotted] & M_{n-2} \arrow[u] \arrow[dl, dotted] & M_{n-1} \arrow[u] \arrow[dl] & M_n \arrow[u] \arrow[dl] \\
 & N_2 \arrow[uu] & N_3 \arrow[u] & \cdots \arrow[u, dotted] & N_{n-2} \arrow[u] & N_{n-1} \arrow[u] & N_n \arrow[u]
\end{tikzcd}
$$

Corollary

Let G be a finite group such that the longest chain of normal subgroups of G has length z, then there are $A, B \in \mathbb{N}$ such that

$$An^{z-1} \leq |\mathfrak{C}_{IS}(G \wr \mathcal{I}_n)| \leq Bn^{2(z-1)}.$$

The number of congruences on $G \wr \mathcal{I}_n$

Theorem (Lima, 1994)

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$$\rho = I^{\star}_{k-1} \cup \overline{\sigma} \cup \chi.$$

Corollary

Let G be a finite group with the length of the longest chain of normal subgroups being z. Then there are $A, B \in \mathbb{N}$ such that

$$An^z \leq |\mathfrak{C}(G \wr \mathcal{I}_n)| \leq Bn^{2z-1}.$$

Thank you for your attention!