Symmetry groups for social preference functions

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A Social Choice question

- Social preference functions

  A committee of $h \geq 2$ individuals needs to order $n \geq 2$ alternatives.

  Every individual expresses his/her preferences by a linear order (a ranking) of the alternatives.

  Those preferences are aggregated into a unique social preference (the final ranking) using a rule.

  Such a rule is called a social preference function (SPF).

- Desirable properties for a SPF
  - Anonymity: the names of individuals are irrelevant
  - Neutrality: the names of alternatives are irrelevant

Problem

- The main rules used to get a decision (Simple majority, Borda count, Minimax, Kemeny rule...) are anonymous and neutral...
- But they are correspondences and not functions!

The severe arithmetical obstructions

Theorem (Bubboloni-Gori, 2014)

Given n alternatives and h individuals,

- there exists an anonymous and neutral SPF if and only if

$$\gcd(h, n!) = 1 \qquad (\star)$$

- When $(\star)$ does not hold, a SPF is only partially anonymous and neutral

Kelly open questions

Posed by the mathematical economist J. S. Kelly in Conjectures and unsolved problems (1991) and remained unsolved

Problems

Given a desired level of anonymity or neutrality, when is it possible to get a SPF having such a level of anonymity or neutrality?

The model

- $N := \{1, \ldots, n\}$, the set of alternatives , $n \geq 2$
- $H := \{1, \ldots, h\}$, the set of individuals, $h \geq 2$

- $\mathcal{L}(N)$, the set of linear orders on N (complete, transitive, antisymmetric relations)

- For $i \in H$, his/her preferences are given by $p_i = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathcal{L}(N)$

Preferences as permutations

$\mathcal{L}(N)$ is identifiable with the symmetric group $S_n = Sym(N)$

$$\psi \in S_n \quad \text{is identified with} \quad \begin{bmatrix} \psi(1) \\ \psi(2) \\ \vdots \\ \psi(n) \end{bmatrix} \in \mathcal{L}(N)$$

- The ordered list $p = (p_i)_{i=1}^h$ is called a preference profile.
- The set of profiles is denoted by $\mathcal{P}$ and is identified with S_n^h

The Group Action

Consider the group $G := S_h \times S_n$.

Given $p \in \mathcal{P}$ and $(\varphi, \psi) \in G$, $p^{(\varphi,\psi)}$ is the profile defined by

$$p_i^{(\varphi,\psi)} := \psi p_{\varphi^{-1}(i)}, \quad \forall i \in H.$$

1. Individual i is renamed $\varphi(i)$
2. Alternative x is renamed $\psi(x)$
3. The map $p \mapsto p^{(\varphi,\psi)}$ defines an action of G on $\mathcal{P}$

Example

- h=5, n=3, $\varphi = (134)(25) \in S_5$, $\psi = (12) \in S_3$ and

$$p = \begin{bmatrix} 1 & 2 & 3 & 1 & 3 \\ 3 & 3 & 1 & 3 & 1 \\ 2 & 1 & 2 & 2 & 2 \end{bmatrix}$$

$$\Downarrow$$

$$p^{(\varphi,id)} = \begin{bmatrix} 1 & 3 & 1 & 3 & 2 \\ 3 & 1 & 3 & 1 & 3 \\ 2 & 2 & 2 & 2 & 1 \end{bmatrix}, \qquad p^{(id,\psi)} = \begin{bmatrix} 2 & 1 & 3 & 2 & 3 \\ 3 & 3 & 2 & 3 & 2 \\ 1 & 2 & 1 & 1 & 1 \end{bmatrix}$$

$$p^{(\varphi,\psi)} = \begin{bmatrix} 2 & 3 & 2 & 3 & 1 \\ 3 & 2 & 3 & 2 & 3 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

SPFs and symmetry

- A SPF is a map $F : \mathcal{P} \to S_n$.

  $\mathcal{F} :=$ the set of the SPFs

- Let $U \leq G$. We say that a SPF F is U-symmetric if

$$F\left(p^{(\varphi,\psi)}\right) = \psi F(p), \quad \forall p \in \mathcal{P},\ \forall (\varphi,\psi) \in U.$$

  $\mathcal{F}^U :=$ the set of U-symmetric SPFs

- F is anonymous $\iff S_h \times \{id\}$-symmetric
- F is neutral $\iff \{id\} \times S_n$-symmetric
- F is anomymous and neutral $\iff S_h \times S_n$-symmetric

Regular Groups

Problema

Given $U \leq G$, under which condition $\mathcal{F}^U \neq \varnothing$?

Define $U \leq G$ regular if, for every $p \in \mathcal{P}$,

$$\mathrm{Stab}_U(p) \leq S_h \times \{id\}$$

Theorem 1 (Bubboloni-Gori, 2015)

Let $U \leq G$. Then $\mathcal{F}^U \neq \varnothing \iff U$ is regular

Characterization of regular groups

$T(\varphi) :=$ list of sizes of the orbits of $\varphi \in S_h$ on $H = \{1, \ldots, h\}$

Theorem 2 (Bubboloni-Gori, 2015)

Let $U \leq G = S_h \times S_n$. The following conditions are equivalent:

a) U is regular

b) If $(\varphi, \psi) \in U$, then $\forall \pi$ prime number,

$$|\psi|_\pi = \pi^a > 1 \Longrightarrow \pi^a \nmid \gcd(T(\varphi))$$

- $V \times \{id\}$, for $V \leq S_h$, is regular
- $\{id\} \times W$, for $W \leq S_n$, is regular
- $S_h \times W$, for $W \leq S_n$, is regular $\Longleftrightarrow \gcd(h, |W|) = 1$

$$\Downarrow$$

For a SPF: easy to get only anonymity or only neutrality, difficult to get both!

Anonymity, neutrality and symmetry groups for a SPF

Let $F \in \mathcal{F}$. Define

1. The Anonymity group of F

$$G_1(F) := \{(\varphi, id) \in G : F(p^{(\varphi, id)}) = F(p),\ \forall p \in \mathcal{P}\}$$

(Kelly)

2. The Neutrality group of F

$$G_2(F) := \{(id, \psi) \in G : F(p^{(id, \psi)}) = \psi F(p),\ \forall p \in \mathcal{P}\}$$

(Kelly)

3. The Symmetry group of F

$$G(F) := \{(\varphi, \psi) \in G : F(p^{(\varphi, \psi)}) = \psi F(p),\ \forall p \in \mathcal{P}\}$$

(New)

- There are links among those concepts but not the trivially expected ones!
- Surely

$$G(F) \geq G_1(F) \times G_2(F)$$

but, generally,

$$G(F) \neq G_1(F) \times G_2(F)$$

$$\Downarrow$$

Symmetry $\neq$ Anonymity $\times$ Neutrality

Anonymity, neutrality and symmetry groups

Definitions

Let $U \leq G = S_h \times S_n$. U is called

1. an Anonymity group if there exists $F \in \mathcal{F}$ such that $G_1(F) = U$
2. a Neutrality group if there exists $F \in \mathcal{F}$ such that $G_2(F) = U$
3. a Symmetry group if there exists $F \in \mathcal{F}$ such that $G(F) = U$

Three problems

1. Which $U \leq G = S_h \times \{id\}$ are anonymity groups? (Kelly)
2. Which $U \leq \{id\} \times S_n$ are neutrality groups? (Kelly)
3. Which $U \leq S_h \times S_n$ are symmetry groups? (New)

- Example: G is a symmetry group $\Leftrightarrow \gcd(h, n!) = 1$

The heart of the matter

- An easy consequence of Theorem 1

Proposition

Let $U \leq G$.

- If U is a symmetry group, then U is regular
- If U is a maximal regular subgroup, then U is a symmetry group

Why U regular $\not\Rightarrow$ U a symmetry group?

U regular $\Rightarrow \mathcal{F}^U \neq \varnothing$, by Theorem 1.

For every $F \in \mathcal{F}^U$, you have $G(F) \geq U$.

You do not know if, for some F, you have $G(F) = U$.

- Similar obstructions appear for anonymity and neutrality

Neutrality groups

Theorem

For every h and n, every $U \leq \{id\} \times S_n$ is a neutrality group

Corollary

Let $W \leq S_n$. If $S_h \times W$ is regular, then it is a symmetry group

An interesting fact

- U neutrality group $\quad \not\Longrightarrow \quad U$ symmetry group
- Example: $\{id\} \times S_2$ is a neutrality group but not a symmetry group when $h = 3$ or $h = 4$.

  Open: What does it happen for $h \geq 5$?

Anonymity and symmetry groups

Proposition

Let $U \leq S_h \times \{id\}$. The following facts are equivalent:

i) U is an anonymity group

ii) U is an symmetry group

- ii)$\Rightarrow$ i) is trivial
- i)$\Rightarrow$ ii) is deep

$$\Downarrow$$

Anonimity groups $=$ Symmetry groups included in $S_h \times \{id\}$

$$\Downarrow$$

Sufficient conditions for anonymity work also for symmetry

Necessary conditions for symmetry work also for anonymity

A sufficient condition for anonymity

Theorem

Let $U \leq S_h \times \{id\}$. If there exists $p \in \mathcal{P}$, such that

$$\mathrm{Stab}_{S_h \times \{id\}}(p) \leq U \quad (*),$$

then U is an anonymity group

Corollary

i) If $H_1, \ldots, H_k$ is a partition of H with $1 \leq k \leq n!$, then

$$[\times_{i=1}^{k} \mathrm{Sym}(H_i)] \times \{id\}$$

is an anonymity group

ii) If $h \leq n!$, then every $U \leq S_h \times \{id\}$ is an anonymity group

Proof: i) satisfy $(*)$ choosing a profile p with $p_i = p_j$ if and only if $i, j \in H_s$ for some $1 \leq s \leq k$; ii) use the partition of H by singletons.

Boolean functions

Case $n = 2$

Here $\mathcal{P} = \{id, (12)\}^h$.

Write $0 := id$, $1 := (12)$ to get $\mathcal{P} = \{0, 1\}^h$.

The action of $V \times \{id\} \leq S_h \times S_2$, on $\mathcal{P}$ only changes the order of $0, 1$ in $p \in \mathcal{P}$.

A SPF is just a boolean function

$$F : \{0, 1\}^h \to \{0, 1\}$$

- Recall that V is called 2-representable if there exists a boolean function F such that

$$\{\varphi \in S_h : F(x_{\varphi^{-1}(1)}, \ldots, x_{\varphi^{-1}(h)}) = F(x), \ \ \forall x \in \{0, 1\}^h\} = V$$

- So V is 2-representable $\iff$ $V \times \{id\}$ is an anonymity group

A necessary condition for symmetry

- A concept from permutation group theory

Definition

Let $U, V \leq G$. We write

- $U \leq_{\mathcal{P}} V \quad \text{if} \quad p^U \subseteq p^V, \quad \forall p \in \mathcal{P}$
- $U \cong_{\mathcal{P}} V \quad \text{if} \quad p^U = p^V, \quad \forall p \in \mathcal{P}$

If $U \cong_{\mathcal{P}} V$ we say that U and V are orbit equivalent

- A crucial fact

Theorem

Let $U, V \leq G$ such that $\langle U, V \rangle$ is regular. Then

- $U \leq_{\mathcal{P}} V \quad \iff \quad \mathcal{F}^V \subseteq \mathcal{F}^U$
- $U \cong_{\mathcal{P}} V \quad \iff \quad \mathcal{F}^V = \mathcal{F}^U$

The group $O(V)$

Definition

Let $V \leq G$ be regular. Define

$$O(V) := \langle U \leq G : \; U \leq_{\mathcal{P}} V, \; \langle U, V \rangle \text{ is regular} \rangle$$

Theorem

Let $V \leq G$ be regular. Then

- $V \leq O(V) = \bigcap_{F \in \mathcal{F}^V} G(F)$
- $O(V)$ is the greatest regular subgroup of G orbit equivalent to V

Corollary- A necessary condition for symmetry

Let $V \leq G$ be regular. If V is a symmetry group then $O(V) = V$

Corollary- A necessary condition for 2-representability

Let $V \leq S_h$. If V is 2-representable then $O(V \times \{id\}) = V \times \{id\}$

Examples

- $O(A_h \times \{id\}) = \begin{cases} S_h \times \{id\} & \text{if} \quad h > n! \\ A_h \times \{id\} & \text{if} \quad h \leq n! \end{cases}$
- For $h \geq 4$, $O(\langle(1234)\rangle \times \{id\}) = \langle(1234), (13)\rangle \times \{id\} \simeq D_8$

Consequence

$A_h \times \{id\}$ is an anonymity group $\iff h \leq n!$

A question

$$O(V) = V \quad \Longrightarrow \quad V \text{ is a symmetry group?} \quad \text{No}$$

Proposition

Let $V \leq G$, where one of the following cases hold

i) $G = S_4 \times S_2$ and $V = K \times \{id\}$, with $K \leq S_4$ the Klein 4-group

ii) $G = S_3 \times S_2$ or $G = S_4 \times S_2$ and $V = \{id\} \times S_2$

Then:

1. $O(V) = V$
2. V is not a symmetry group

- $K \times \{id\}$ is a key example for the boolean function 2-representability
- $K \times \{id\}$ is the unique not symmetric group among the $V \leq S_h \times \{id\}$ having $O(V) = V$, within many classes of permutation groups (M. Grech and A. Kisielewicz (1998-2014))

The Conjecture

- All the examples we know are about $n = 2$...

Conjecture for sandgal-19

Let $n \geq 3$ and $V \leq G$ be regular. Then

$$V \text{ is a symmetry group} \quad \Longleftrightarrow \quad O(V) = V$$

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