The Howson property for one-sided ideals of semigroups

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Right (left) ideal Howson semigroups

Definition

An algebra is said to have the **Howson property** if the intersection of any two finitely generated subalgebras is also finitely generated.

By considering a semigroup S as being a right (left) S-act we define the Howson property within the context of one-sided ideals.

Definition

A semigroup S is called **right (left) ideal Howson** if the intersection of any two finitely generated right (left) ideals of S is finitely generated.

It is sufficient to check that the intersections of principal right (left) ideals are finitely generated to determine that S is right (left) ideal Howson.

Right (left) ultra Howson semigroups

Definition

A semigroup S is **right** (**left**) **ultra Howson** if S is right ideal (left ideal) Howson and there exists $a, b \in S$ such that $aS^1 \cap bS^1$ ($S^1a \cap S^1b$) is not principal.

Say a right (left) ideal I of a semigroup S is **exactly** n**-generated** for $n \in \mathbb{N}^0$ if I is generated by n elements (where no $n-1$ elements will suffice).

Definition

A semigroup S is **right** (**left**) n**-ultra Howson** if S is right (left) ideal Howson and there exists $a, b \in S$ such that $aS^1 \cap bS^1$ ($S^1a \cap S^1b$) is exactly n-generated.

Motivation

R. Exel and B. Steinberg were interested in the construction of $\mathbb{C}^*$-algebras from the inverse hulls of certain semigroups. In particular these semigroups are 0-left cancellative and finitely aligned.

Definition

A semigroup S is finitely aligned if for all $a, b \in S$ there exists $r_1, \ldots, r_n \in S$ such that

$$Sa \cap Sb = \bigcup S^1 r_i.$$

One way in which these semigroups can be obtained is from higher rank graphs. Steinberg asked whether or not there exist 'natural' examples of right n-ultra Howson semigroups.

Examples

Many familiar examples of semigroups are right and left ideal Howson.

- Groups
- Inverse semigroups
- Null semigroups
- Right/left zero semigroups
- Free semigroups
- Full transformation monoids

Additionally any finite semigroup is both right and left ideal Howson.

Closure properties

Theorem

The set of right (left) ideal Howson semigroups is not closed under direct products.

Sketch

Know $S = (\mathbb{N}, +)$ is right ideal Howson and we let $T = S \times S$. Consider the intersection I given by

$$I = (1,1)T^1 \cap (1,2)T^1.$$

Suppose I is f.g. by say $I = \bigcup_{i=1}^{n}(a_i, b_i)T^1$ and notice

$$J = \{(a,3) : 2 \leq a \leq n+2\} \subset I.$$

We must have some $(h,3) \neq (k,3) \in J$ in some $(a_i, b_i)T^1$. This leads to a contradiction.

Closure properties

Theorem

The set of right (left) ideal Howson semigroups is closed under free products.

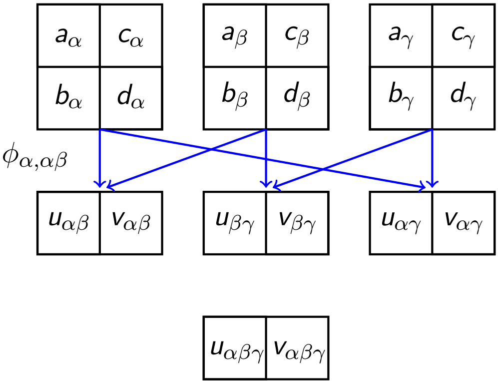
Normal bands

a_α	c_α		a_β	c_β		a_γ	c_γ
b_α	d_α		b_β	d_β		b_γ	d_γ

$u_{\alpha\beta}$	$v_{\alpha\beta}$		$u_{\beta\gamma}$	$v_{\beta\gamma}$		$u_{\alpha\gamma}$	$v_{\alpha\gamma}$

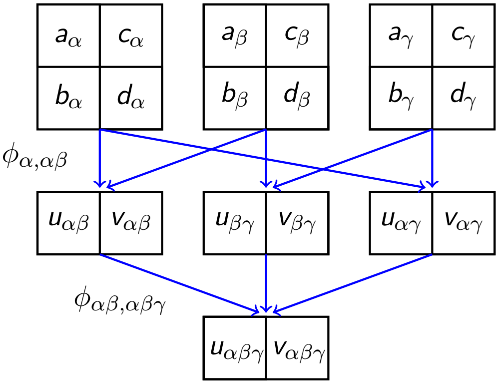
$u_{\alpha\beta\gamma}$	$v_{\alpha\beta\gamma}$

Normal bands



$$s_\alpha \phi_{\alpha,\alpha\beta} = \begin{cases} u_{\alpha\beta} & s_\alpha \in \{a_\alpha, b_\alpha\} \\ v_{\alpha\beta} & s_\alpha \in \{c_\alpha, d_\alpha\} \end{cases}$$

Normal bands



$$s_\alpha \phi_{\alpha,\alpha\beta} = \begin{cases} u_{\alpha\beta} & s_\alpha \in \{a_\alpha, b_\alpha\} \\ v_{\alpha\beta} & s_\alpha \in \{c_\alpha, d_\alpha\} \end{cases}$$

$$u_{\alpha\beta,\alpha\beta\gamma}\phi_{\alpha\beta,\alpha\beta\gamma} = u_{\alpha\beta\gamma} \text{ and } v_{\alpha\beta,\alpha\beta\gamma}\phi_{\alpha\beta,\alpha\beta\gamma} = v_{\alpha\beta\gamma}$$

Bands

In fact, we were able to determine which varieties of bands will contain such an example.

Definition

A **right regular band** S is a band such that $aba = ba$ for all $a, b \in S$.

Theorem

Let $\mathscr{V}$ be a variety of bands not contained in the variety $\mathscr{RR}$ of right regular bands. Then for any $n \in \mathbb{N}$, with $n \geq 2$, the variety $\mathscr{V}$ contains a band B that is right n-ultra Howson.

Let B be a band in a variety $\mathscr{V}$ contained in $\mathscr{RR}$. For all $a, b \in B$ we have $ab \in aB^1 \cap bB^1$ and if $u \in aB^1 \cap bB^1$ then $u = abu \in abB^1$. Thus $aB^1 \cap bB^1$ is principal.

Right (left) ideal Howson semigroup presentations

Definition

A monoid S is **right coherent** if every finitely generated subact of every finitely presented right S-act is finitely presented.

Theorem (VG, 1992)

A monoid M is right coherent if and only if for any finitely generated right congruence θ on M and any $a, b \in M$ we have $\{(u, v) \in M \times M : [a]u = [a]v\}$ is finitely generated as a right congruence on M and $[a]M \cap [b]M$ is finitely generated as an M-subact of M/θ.

Proposition

If S is a semigroup and S^1 is a right coherent then S is right ideal Howson.

Right (left) ideal Howson semigroup presentations

Proposition

Let S_τ have commutative semigroup presentation

$$\langle X : \tau \rangle$$

where τ is finite. Then S_τ is right and left ideal Howson.

By Rédei's Theorem any congruence on a finitely generated commutative monoid is finitely generated.

Corollary

Any finitely generated commutative semigroup is right and left ideal Howson.

Right (left) ideal Howson semigroup presentations

Example

Let S be given by the commutative semigroup presentation

$$\langle a, b, u_i, v_i \, (i \in \mathbb{N}) : au_i = bv_i \, (i \in \mathbb{N})\rangle.$$

Then S is not right or left ideal Howson.

Example

Let S be given by the semigroup presentation

$$\langle a, b, c, d, p, q, u, v : auvc = bpqd, au = ua, ub = bp, uv = u^2v^2\rangle.$$

Then S is not right ideal Howson.

Right (left) ultra Howson semigroup presentations

For some fixed $n \in \mathbb{N}$ where $n \geq 2$, let X be the non-empty set

$$X = \{a, b, u_i, v_i : 1 \leq i \leq n\}$$

and let ρ and λ be subsets of $X^+ \times X^+$ given by

$$\rho = \{(au_i, bv_i) : 1 \leq i \leq n\} \text{ and } \lambda = \{(u_i a, v_i b) : 1 \leq i \leq n\}.$$

Theorem

For all $n \in \mathbb{N}$ where $n \geq 2$, the semigroup S_ρ with presentation $\langle X : \rho \rangle$ is right $n-$ultra Howson.

Moreover we showed that this semigroup is cancellative.

Right and left ultra Howson semigroup presentations

Theorem

For all $n \in \mathbb{N}$ where $n \geq 2$, the semigroup $\bar{S}_\rho$ with commutative presentation $\langle X : \rho \rangle$ is right and left n-ultra Howson.

Let σ be the subset of $X^+ \times X^+$ given by

$$\sigma = \rho \cup \{(u_i v_j, u_j v_i) : 1 \leq i \leq n\}.$$

Theorem

For all $n \in \mathbb{N}$ where $n \geq 2$, the semigroup $\bar{S}_\sigma$ with commutative presentation $\langle X : \sigma \rangle$ is right and left n-ultra Howson.

We showed that this semigroup is cancellative.

Universal property

Proposition

Let S be a semigroup such that S contains two principal right ideals αS and βS such that $\alpha S \cap \beta S$ has exactly n generators $\alpha\gamma_1 = \beta\delta_1, \ldots, \alpha\gamma_n = \beta\delta_n$. Then there is a homomorphism $\theta : S_\rho \to S$ such that $[a]\theta = \alpha, [b]\theta = \beta, [u_i]\theta = \gamma_i$ and $[v_i]\theta = \delta_i$ for all $1 \leq i \leq n$.

Proposition

Let S be a commutative (commutative and cancellative) semigroup such that S contains two principal right ideals αS and βS such that $\alpha S \cap \beta S$ has exactly n generators $\alpha\gamma_1 = \beta\delta_1, \ldots, \alpha\gamma_n = \beta\delta_n$. Then there is a homomorphism $\theta : \bar{S}_\rho \to S$ ($\bar{S}_\sigma \to S$) such that $[a]\theta = \alpha, [b]\theta = \beta, [u_i]\theta = \gamma_i$ and $[v_i]\theta = \delta_i$ for all $1 \leq i \leq n$.

Further thoughts

- Are there any more closure properties for the set of right ideal Howson semigroups?
- Are there any more closure properties for the set of right ultra Howson semigroups?
- Can we find any other semigroup varieties that contain examples of right ultra Howson semigroups?
- Are there any semigroup presentations that determine a non-commutative right and left ultra Howson semigroup?
- Is there any significance of the semigroups S_ρ, S_λ, $\bar{S}_\rho$ and $\bar{S}_\eta$ to the construction of $\mathbb{C}^*$-algebras?

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