On Cayley graphs of basic algebraic structures

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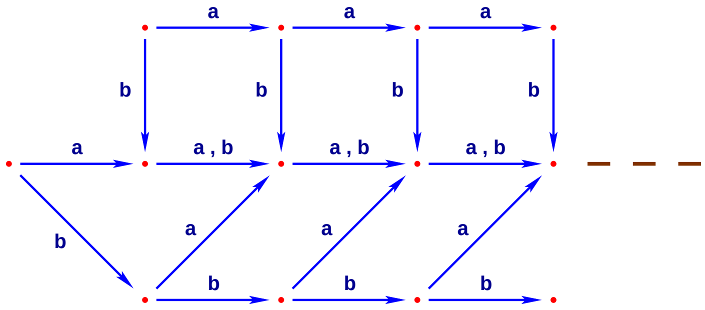
Graph-theoretic characterizations

for the Cayley graphs of

- monoids and commutative monoids
- semigroups and semilattices
- groups and abelian groups



Cayley graph of a commutative semigroup

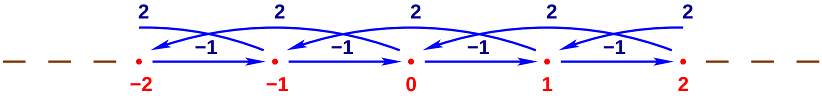
Magma $(\mathsf{M},\cdot)$ and any subset S of M

set A of labels and an injection $[\![\ ]\!] : \mathsf{S} \rightarrow \mathsf{A}$

Generalized Cayley graph

$$\mathcal{C}[\![\,\mathsf{M},\mathsf{S}\,]\!] \;=\; \{\; \mathsf{m} \xrightarrow{[\![s]\!]} \mathsf{m}\cdot\mathsf{s} \mid \mathsf{m}\in\mathsf{M} \;\wedge\; \mathsf{s}\in\mathsf{S} \;\}$$

For $(\mathbb{Z},-)$ the gen. Cayley graph $\mathcal{C}[\![\,\mathbb{Z}\,,\{-1,2\}\,]\!]$ is



Proposition 1

The generalized Cayley graphs of magmas with a left identity are the deterministic, source-complete graphs with an out-simple vertex.

deterministic : if $s \xleftarrow{a} r \xrightarrow{a} t$ then $s = t$

source-complete : $\forall$ vertex s label a, $s \xrightarrow{a}$

out-simple vertex r : if $r \xrightarrow{a,b}$ then $a = b$

$\Longleftarrow$: G deterministic and source-complete graph

r out-simple vertex

magma $(V_G\,,\cdot)$ with $s \cdot t$ defined by

t if $s = r$

x if $r \xrightarrow{a} t$ and $s \xrightarrow{a} x$

r otherwise.

r is a left identity

$S_r = \{\, s \mid r \xrightarrow{a} s \,\}$ set of successors of r

$G = \mathcal{C}[\![\, V_G\,, S_r \,]\!]$ with $[\![\, s \,]\!] = a$ for $r \xrightarrow{a} s$

Proposition 2

The generalized Cayley graphs of magmas with an identity are the deterministic, source-complete graphs with an out-simple loop-propagating vertex.

r loop-propagating : if $r \xrightarrow{a} r$ then $s \xrightarrow{a} s \ \forall \ s$

$\Longleftarrow$: We define $s \cdot t$ by

t if $s = r$

s if $t = r$

x if $r \xrightarrow{a} t$ and $s \xrightarrow{a} x$

r otherwise.

Proposition 3

The generalized Cayley graphs of commutative magmas with an identity are the deterministic, source-complete graphs with an out-simple loop-propagating and locally commutative vertex.

r locally commutative : if $r \xrightarrow{ab} s$ then $r \xrightarrow{ba} s$

$\Longleftarrow$: We define $s \cdot t$ by r an identity, otherwise

x if $r \xrightarrow{a} t$ and $s \xrightarrow{a} x$

y if $r \xrightarrow{b} s$ and $t \xrightarrow{b} y$

r otherwise.

Cayley graph of a monoid $(M,\cdot)$

$\mathcal{C}[\![M,S]\!]$ for M generated by S :

$$M = \{ s_1 \cdot \ldots \cdot s_n \mid n \geq 0 \text{ and } s_1,\ldots,s_n \in S \}$$

Theorem 1

The Cayley graphs of (*resp. commutative*) *monoids are the deterministic, source-complete graphs with an out-simple root which is propagating* (*resp. and locally commutative*).

r propagating : if $r \xrightarrow{u,v}$ then $s \xrightarrow{u,v}$ for any s

$\Longleftarrow$: G deterministic and source-complete graph

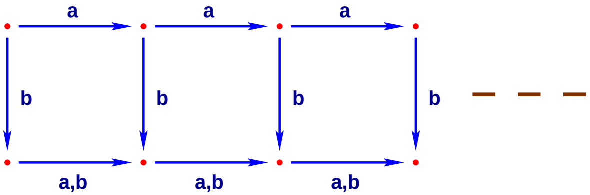
root r out-simple and propagating vertex

We define $s \cdot t$ by

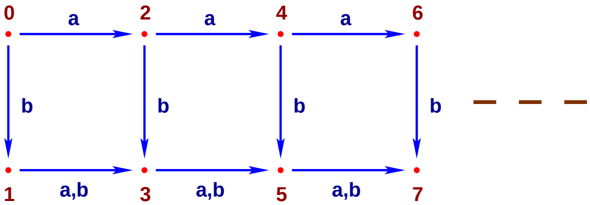
$$s \xrightarrow{u} s \cdot t \quad \text{if} \quad r \xrightarrow{u} t \ \text{ with } u \in A^*$$

$(V_G , \cdot)$ monoid generated by S_r

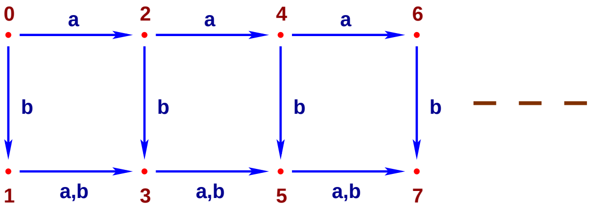
$G = \mathcal{C}[\![V_G , S_r]\!]$ with $[\![s]\!] = a$ for $r \xrightarrow{a} s$



Cayley graph of a commutative monoid



Cayley graph of a commutative monoid



Cayley graph of a commutative monoid

$$m \cdot n = \begin{cases} m + n & \text{if } m \text{ or } n \text{ is even} \\ m + n + 1 & \text{if } m \text{ and } n \text{ are odd} \end{cases}$$

So $\cdot$ is associative, commutative of identity 0

Cayley graph of a semigroup $(M,\cdot)$

$\mathcal{C}[\![M,S]\!]$ for M generated by S :

$$M = \{ s_1 \cdot \ldots \cdot s_n \mid n > 0 \text{ and } s_1,\ldots,s_n \in S \}$$

Corollary 1

Cayley graph G *of a semigroup* (*resp. commutative*) :

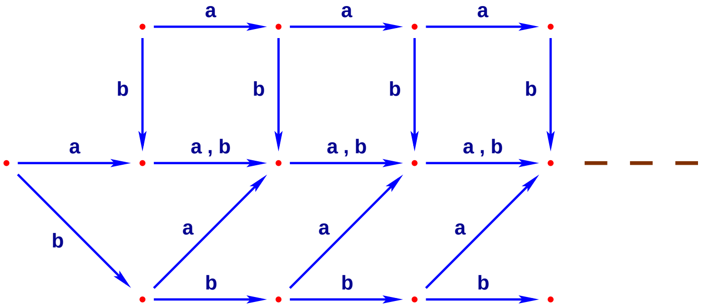
G *deterministic and* $\exists\ i : A_G \longrightarrow V_G$ *injective s.t.*

V_G *accessible from* $i(A_G)$

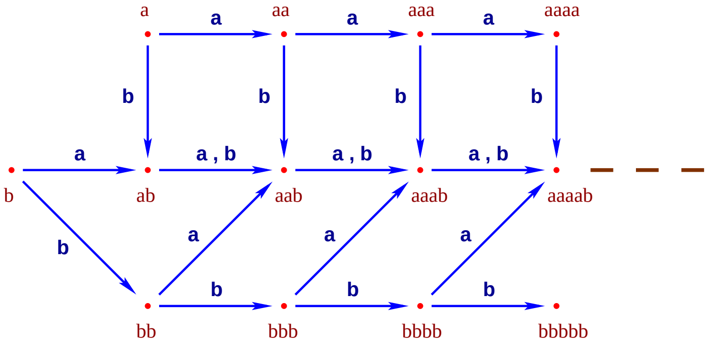
if $i(a) \xrightarrow{u} \xleftarrow{v} i(b)$ *then* $s \xrightarrow{au,bv}$

for any vertex s *label* a,b *label word* u,v

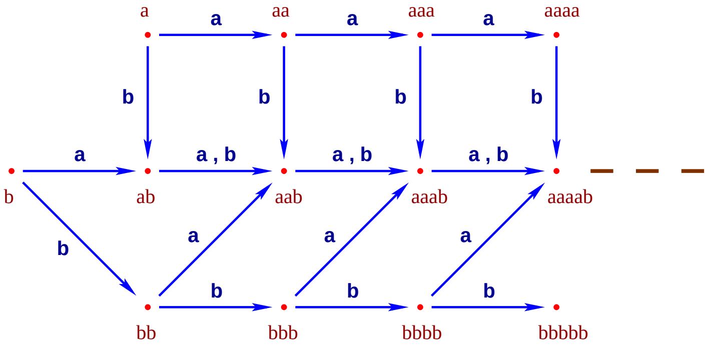
(*resp* $i(a) \xrightarrow{b} \xleftarrow{a} i(b)$ *for any label* a,b).



Cayley graph of a commutative semigroup



Cayley graph of a commutative semigroup



Cayley graph of a commutative semigroup

·	a^n	$a^j b$	b^q
a^m	a^{m+n}	$a^{m+j} b$	$a^{m+q-1} b$
$a^i b$	$a^{i+n} b$	$a^{i+j+1} b$	$a^{i+q} b$
b^p	$a^{p+n-1} b$	$a^{p+j} b$	b^{p+q}

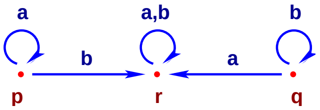
Semilattice $(M,\cdot)$

commutative semigroup with $\cdot$ idempotent

Corollary 2

Cayley graph G *of a semilattice*:

$\ldots$ *if* $i(a) \xrightarrow{u} s$ *then* $s \xrightarrow{au} s$ *for any* $\ldots$



$\cdot$	p	q	r
p	p	r	r
q	r	q	r
r	r	r	r

$\mathcal{C}[\![\,\{p,q,r\},\{p,q\}\,]\!]$ with $[\![\,p\,]\!] = a$; $[\![\,q\,]\!] = b$

Cayley graph of a group $(M,\cdot)$

$\mathcal{C}[\![M,S]\!]$ for M generated by S :

$$M = \{ s_1 \cdot \ldots \cdot s_n \mid n \geq 0 \text{ and } s_1,\ldots,s_n \in S \cup S^{-1} \}$$

Theorem ICGT 2018

The Cayley graphs of groups are the connected, simple, deterministic and co-deterministic vertex-transitive graphs.

G vertex-transitive : all the vertices are isomorphic

G simple : if $s \xrightarrow{a,b} t$ then $a = b$

G co-deterministic : $\overline{G}$ deterministic

$$\overline{G} = \{ t \xrightarrow{\bar{a}} s \mid s \xrightarrow{a} t \} \text{ with } \{ \bar{a} \mid a \in A \} \text{ copy of } A$$

Theorem 2

Cayley graph G *of a group* (*resp. abelian*) :

G *connected, deterministic and co-deterministic with a(ny) vertex which is source and target-complete, in and out-simple, and chain-propagating* (*resp. and locally commutative*).

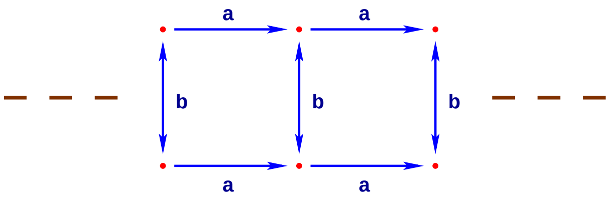
r chain-propagating : r propagating for $G \cup \overline{G}$

r target-complete : r source-complete for $\overline{G}$

r in-simple : r out-simple for $\overline{G}$

$\Longleftarrow$: We define $s \cdot t$ in $G \cup \overline{G}$ by

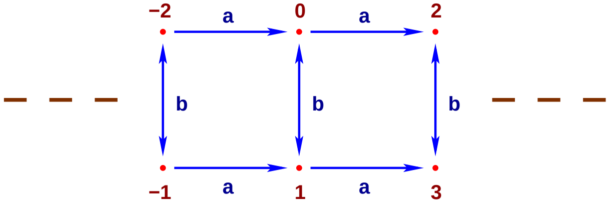
$$s \xrightarrow{u} s \cdot t \quad \text{if} \quad r \xrightarrow{u} t \ \text{ with } u \in (A \cup \overline{A})^*$$



Cayley graph of an abelian group

$\Longleftarrow$: We define $s \cdot t$ in $G \cup \overline{G}$ by

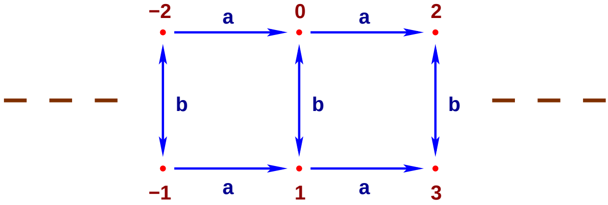
$s \xrightarrow{u} s \cdot t$ if $r \xrightarrow{u} t$ with $u \in (A \cup \overline{A})^*$



Cayley graph of an abelian group

$\Longleftarrow$: We define $s \cdot t$ in $G \cup \overline{G}$ by

$$s \xrightarrow{u} s \cdot t \quad \text{if} \quad r \xrightarrow{u} t \quad \text{with } u \in (A \cup \overline{A})^*$$



Cayley graph of an abelian group

$$m \cdot n = \begin{cases} m + n & \text{if } m \text{ or } n \text{ is even} \\ m + n - 2 & \text{if } m \text{ and } n \text{ are odd} \end{cases}$$

Conclusion

Cayley graphs of semigroups, monoids, groups

... other algebraic structures