On groups satisfying the weak minimal condition on non-core-finite subgroups

work-in-progress by
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SandGAL 2019
Semigroups and Groups, Automata, Logics
Cremona, June 10-13, 2019

1) Groups in which ALL subgroups have some property

Theorem (Dedekind, Baer). In a group G **all the subgroups are normal** if and only if G is **either abelian or hamiltonian,**
i.e. the direct product of a quaternion group of order 8, an elementary abelian 2-group and an abelian group with all its elements of odd order.

There are very many generalizations of normalilty, for exmple:
A group G is called **metahamiltonian** if any non-abelian subgroup is normal.

Theorem (Romails, Sesekin, 1966/9) Let G be any locally (soluble-by-finite) metahamiltonian group, then G' is finite of prime-power order.

Theorem (De Mari and de Giovanni, 2005) Let G be a locally graded metahamiltonian group. Then G'' is abelian.

For more on this subject? see M. De Falco's talk on Wednesday.

2) approximations of normality

B.H. Neuman, 1955

(**an**) H is called almost-normal if $|G : N_G(H)| < \infty$
(AN) All subgroups of the group G are **an** $:=$ almost normal iff G is centre-by-finite.

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(**nn**) H is called nearly-normal if $|H^G : H| < \infty$
NN) All subgroups of the group G are **nn** := nearly normal iff G is finite-by-abelian.

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J.T.Buckley, J.C.Lennox, B.H.Neumann, H.Smith, J.Wiegold, 1995

(**cf**) H is called normal-by-finite (core-finite) subgroup if $|H : H_G| < \infty$.
CF) if all subgroups of the PILF group G are **cf** := core-finite:=normal-by-finite ,
then G is abelian-by-finite.

PILF := Periodic Images are Locally Finite,
Locally Finite:= fintely generated subgroups are finite.
JOB: put all the results in the same framework.

(3) a more general approximation: cn -subgroups

Recall that two subgroups H and K are **commensurable** if their meet $H \cap K$ has finite index in both of them.

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A subgroup H is called a **cn-subgroup if it is commensurable with a normal subgroup** N
(**cn**) there is $N \lhd G$ such that $|HN : (H \cap N)| < \infty$.

This concept generalizes both the standard notions

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Theorem (Bergman, Lenstra, (1989), J. Algebra 127)
A subgroup H of a group G is **commensurable with a normal subgroup**[a] of G
if and only if
$\exists n \in \mathbb{N}$ such that $|H : H^g \cap H| \leq n$ for all $g \in G$. H is *uniformly inert* [b]

[a] they say *close to be normal*

[b] in the infinite alternating group there is an elementary abelian 3-subgroup which is inert but not uniformly inert

Note that if H is **cn**, then H has a subgroup with finite index $(H \cap N)_N$ which is subnormal in G with defect≤ 2, .

Problem: find sufficient conditions for a **cn**-subgroup $H \sim N \lhd G$ to be cf.
Since $|H : H \cap N| < \infty$ we may assume $H = H \cap N \leq N$, we sometimes reduce to:
Problem': find sufficient conditions for a **nn**-subgroup to be **cf**.

Basic cases when **cn** implies **cf**

Let H a **cn**-subgroup of a group G, then H is **cf**, provided one of the following holds.

1) H is abelian-by-finite
2) H is **an** i.e. H has finitely many G-conjugates .
3) H is finitely generated,
4) H has finite Prüfer rank. [a]

[a] A group is said to have finite Prüfer rank r if every finitely generated subgroup can be generated by r elements and r is the least such integer.

5) groups in which all subgroups are cn

B.H. Neuman, 1955

NN) if all subgroups of the group G are **nn** (nearly normal),
then G is **finite-by-abelian** , i.e. the derived subgroup G' is abelian.

Further, if G has the property **PILF** (Periodic Images are Locally Finite), then

J.T.Buckley, J.C.Lennox, B.H.Neumann, H.Smith, J.Wiegold, 1995

CF) if all subgroups of the PILF group G are **cf** (normal-by-finite),
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C.Casolo, U.D., S.Rinauro, J. Algebra 496 (2018)

CN) if all subgroups of the PILF group G are **cn**,, then
- G is finite-by- **CF** (also sufficent)
- G is **NN**-by-finite (not sufficent)

It follows that G is **finite-by-abelian-by-finite**.

Moreover, for each prime there is a soluble p-group in which all subgroup are cn, but the group is neither finite-by-abelian nor abelian-by-finite.

6) Min-∞-$\mathcal{P}$:= the weak minimal condition on $\mathcal{P}$-subgroups.

- A group G is said to have the weak minimal condition on $\mathcal{P}$-subgroups

Min-∞-$\mathcal{P}$

if there are **no** infinite descending chains of $\mathcal{P}$-subgroups of G

$$H_0 \gg H_1 \gg \ldots \gg H_n \gg \ldots$$

(∞) in which all indices $|H_i : H_{i+1}|$ are infinite.

- It is easy to see that a soluble-by-finite Minimax group has Min-∞ (on all subgroups). A group is called **Minimax** if it has a finite series whose factors satisfy Max or Min.

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Several results on generalized soluble groups Min-∞-$\mathcal{P}$ have appeared.

- Baer /Zaitsev (1968) If G is a (locally) soluble group with the weak minimal condition (on all abelian subgroups), then G is a soluble minimax group.

Theorem (Zaicev 1971, Ukrain. Mat. Z. 23)

For locally soluble-by-finite groups, conditions Min-∞,Max-∞, Minimax are equivalent.

7) is Min-∞-$\mathcal{P}$ a real minimal condition in a poset?

Recall that two subgroups H and K are called commusurable if $(H \cap K)|$ has finite index in both of them.
Commensurabilty is an equivalence relation and the **set of commensurability classes** is a POSET.

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FACT

If the property $\mathcal{P}$ is closed either w.r.t.
- intersections of two subgroups or
- commesurability,

the condition Min-∞-$\mathcal{P}$ is indeed **the minimal condition in the poset $\tilde{\mathcal{C}_{\mathcal{P}}}(G)$ of the commensurabilty classes of $\mathcal{P}$-subgroups**.
This applies for
- Min-∞
- Min-∞-n
- Min-∞-sn
- Min-∞-cn
- Min-∞-not-cn
- Min-∞-ab

8) Min-∞-*sn* := the weak minimal condition on (sub)normal-subgroups

- Zaitsev/Baer (1968) If G is a locally soluble group with the weak minimal condition (on all subgroups), then G is a soluble minimax group.

- Kurdachenko (Mat. Zametki 29, 1981): if G is a generalized radical[1] group , then G has Min-∞-*sn* if and only if G is soluble minimax.

- Kurdachenko has studied the weak minimal condition for normal subgroups, Min-∞-*n*. For example (1984) locally nilpotent (non-Dedekind) groups with Min-∞-*n* are hypercentral and soluble but need not be minimax.

[1] a group is called **generalized radical** if it has an ascending series of normal subgroups the factors of which are locally (nilpotent or finite)

9) The DICOTOMY for Min-∞-not-$\mathcal{P}$ when $\mathcal{P}$ is (abelian or normal)

Theorem Phillips and Wilson, J. Algebra 51 (1978)

Let G be a locally (soluble-by-finite) group.

If G satisfies the ..... **minimal condition** on **non-normal non-abelian** subgroups, then

(D) either G has Min..... or each subgroup of G is (abelian or normal),

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Theorem, F. De Mari, Beitr. Algebra Geom., 2019

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Let G be a group with an ascending series with locally (soluble-by-finite) factors.

If G satisfies the **weak minimal** condition on non-normal non-abelian subgroups, then

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We consider the following **dicotomy** (D) for a "generalized" soluble group G:

(D_∞) G has Min-∞-not-$\mathcal{P}$ iff either G is Minimax or all subgroups have $\mathcal{P}$.

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A locally graded[a] group with the minimal condition on non-normal subgroup, then has either the minimal condition or all subgroups are normal.

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L. A. Kurdachenko and V. E. Goretskii, Ukrainian Math. J. 41 (1989)

(D_∞) HOLDS for locally (soluble-by-finite) groups and $\mathcal{P}=$ normality.

L. A. Kurdachenko - H. Smith (1997), Comment. Math. Univ. Carolin. 46 (2005),

(D_∞) HOLDS for generalized radical [a] groups and $\mathcal{P}=$subnormality [b]
(and G results to be soluble-by-finite)

[a] a group is called **generalized radical** if it has an ascending series of normal subgroups the factors of which are locally (nilpotent or finite)

[b] A subgroup H is **subnormal** if there is a finite series $H = H_0 \lhd H_1 \lhd \ldots H_n = G$.

Theorem, Celentani, Leone, Boll. Un. Mat. Ital. A (7) 11 (1997),

A locally graded group satisfying the **minimal condition** on **non-permutable** subgroups either satisfies the minimal condition on arbitrary subgroups or each subgroup is permutable. [a]

[a] for each subgroups $H, K \leq G$, it holds $HK = KH$.

11) the Dicotomy for the Full Minimal condition for modularity/permutability

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Theorem, de Giovanni, Musella, Ricerche Mat. 50 (2001)

A locally graded group satisfying the **minimal condition** on **non-modular** subgroups either satisfies the minimal condition on arbitrary subgroups or each subgroup is modular. [a]

[a] i.e. for each subgroups $X, H, K \leq G$, if $X \leq K$ it holds $\langle X, H\rangle \cap K = \langle X, H \cap K\rangle$

11) the weak minimal condition for non-modular/permutable subgroups

Theorem F. De Mari (2018), Comm. in Algebra, 46

Let G be a group with an ascending series with locally (soluble-by-finite) factors. The following conditions are equivalent:

(i) G satisfies the **weak minimal** condition on **non-modular** subgroups;
(ii) G satisfies the **weak maximal** condition on **non-modular** subgroups;
(iii) each subgroup of G is **either minimax or modular**;
(D) G is **either** a soluble-by-finite **minimax** group **or** each subgroup is **modular** [a]

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Corollary Let G be a group with an ascending series with locally (soluble-by-finite) factors. The following conditions are equivalent:

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(2) G satisfies the **weak maximal** condition on **non-permutable** subgroups;

(3) any subgroup of G is either **minimax** or **permutable**;

(D) G is either a soluble-by-finite **minimax** group or each subgroup is **permutable**. [a]

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12) approximations of normality: Dicotomy for the Full Minimal condition

Theorem, Kurdachenko, Pylaev, Ukrainian Math. J. 40 (1988)

Let G be a group with Min-non- **an** and some non- **an** subgroup.

D) if G is locally finite, then G has Min

*) if G is non-periodic, then G has a finite normal subgroup E such that $G/E = \bar{J} \times \bar{K}$ where $\bar{J}$ is the direct product of finitely many Prüfer groups and $\bar{K}$ is the extension of a free abelian group of rank $p-1$ by an automorphism of order p acting rationally irreducibly.

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Theorem Franciosi, de Giovanni, Korea 1995

Let G be a group with Min-non- **nn** and some non- **nn** subgroup.

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*) if G is non-periodic group, then G has a finite normal subgroup E s.t. $G/E = \bar{J} \times \bar{D}$ where $\bar{J}$ is the direct product of finitely many Prüfer groups and $\bar{D}$ is infinite dihedral.

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Let now G be group whose Periodic Images are Locally Finite, PILF.

Theorem, Franciosi, de Giovanni, Proc. Amer. Math. Soc. 125 (1997)

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D) If G has Min-non- **cf** and some non- **cf** subgroup, then G has Min,

Theorem, UD and S.Rinauro, Adv. Group Theory Appl. 7, (2019)

D) If G has Min-non- **cn** and some non- **cn** subgroup, then G has Min.

Theorem (Cutolo-Kurdachenko, 1993)

Let G be a group with an ascending series with locally (soluble-by-finite) factors. Then

- G has Min-∞-non- **an** subgroups iff all non-minimax subgroups are **an**

or, equivalently, one of the following 1,2,3 holds :

1) G is Minimax
2) all subgroups are **an** (i.e. $G/Z(G)$ is finite)
3) G has a finite normal subgroup E such that $G/E = P \times N$ where
 i) P is the product of finitely many Prüfer groups,
 ii) N is class 2 nilpotent torsion-free with finite rank and $N/Z(N)$ finitely generated
 iii) N' is contained in all pure subgroups of $Z(N)$ wich are not minimax.

Notes:

- when G is periodic (=locally finite), then case (3) does not accour and (1) just means that G is Chernikov (has Min)
- $P = tor(G)$
- N is non minimax but of finite rank finite rank.
- N' is finitely generated free abelian.
- in case (3) $Z(G)$ is minimax-by-(torsion-free with rank 1).

Theorem (De Mari, Bull. UMI 10-B, 2007)
Let G be a group with an ascending series with locally (soluble-by-finite) factors.
Then
- G has Min-∞-non- **nn** iff all non-minimax subgroups are **nn**
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Notes: **Notes**
- If $E \lhd G$ is finite, then G has Min-∞-non- **nn** if and only if G/E has.
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Let $E \lhd G$ be finite. Then G has Min-∞-non- **cn** if and only if G/E has.

Let $E \triangleleft G$ be finite. Then G has Min-∞-non- **cn** if and only if G/E has.

Theorem. Let G be a Min-∞-non- **cn** group.

D) if G is **locally finite**, then G is either a (soluble-by-finite) CN-group or a Černikov group.

ii) if $G$is **generalized radical** [a], then G is **soluble-by-finite**.

[a] that is it has an ascending series with locally (nilpotent or finite) factors

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Theorem. Let G be a soluble-by-finite group.

- G satisfies Min-∞-non- **cn** iff **all non-minimax subgroups of G are cn-subgroups**.

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Theorem. Let G be a soluble-by-finite group.

- G satisfies Min-∞-non- **cn** iff **all non-minimax subgroups of G are cn-subgroups**.

Theorem. Let G be a soluble-by-finite group satisfying Min-∞-non- **cn**

If G is neither (1)=minimax nor (2)=CN, then
G has **finite rank** and G has a normal subgroup of finite index satisfying Min-∞-non- **nn**.
(of type (3) above).

13') the WEAK Minimal condition on non- cn subgroups

Theorem

Let G be a group with an ascending series with locally (soluble-by-finite) factors.
Then

- G has Min-∞-non- **cn** iff all non-minimax subgroups are **cn**

and necessarily we have that one of the following holds

1) G is Minimax

2) all subgroups are **cn**

3) there is (normal) subgroup G_0 with finite index and a finite normal subgroup E such that $G_0/E = P \times N$ where

i) P is the product of finitely many Prüfer groups,

ii) N is class 2 nilpotent torsion-free with finite rank and $N/Z(N)$ finitely generated

iii) N' is contained in all pure subgroups of $Z(N)$ wich are not minimax.

THANK YOU

for your attention

looking for a sufficient condition

Let G be a soluble-by-finite group satisfying the weak minimal condition on non-cn-subgroups which is neither minimax nor a CN-group.
Then modulo a finite normal subgroup F, the group $G := G/F$ has normal subgroups P, N, X such that
1) $\langle P, N \rangle = P \times N$ has finite index in G
2) P is direct product of finitely many Prufer groups and each subgroup is normal in G
3.1) N is class 2 nilpotent torsion-free with finite rank and $N/Z(N)$ finitely generated
3.2) N' is contained in all pure subgroups of $Z(N)$ wich are not minimax.
4) $N' \leq X \leq N$
5) X is a finitely generated free abelian group
6) G/X is a periodic CN-group
7) X/N'....

proof: let $G_0 \leq_f G$ as in the prevoius statement and $E \lhd G_0$ finite such that .... Then E is *an* in G, so that by Dicman's Lemma $E := E^G$ is finite and we can factor it out and assume $G_0 = P \times N$ as in ....
Since G is not minimax and P is, we have that N is not minimax and by.... N is a *cn* subgroup.
Since G has finite rank, then N is *cf* and we may assume ***?? $N := N_G\ nG$ and N retains the same structure??? and (1) and (3) hold.
On the other hand, since N is not minimax, each subgroup of $P \simeq_G G_0/N$ is cn in G, therefore by Lemma 2.3.i of [CDR] G we have (2).
then take X/N' to be torsion free maximal in N/N'. Since N/X is periodic with infinitely many

THANK YOU

for your attention