Additive semigroups of integers Embedding dimension of numerical semigroups

Violeta Angelkovska*, Donco Dimovski**

*FON University, Skopje; **Ss Ciril and Methodius University, Skopje

SANDGAL 2019

This talk is motivated by the results, stated in the papers:

[1] Д. Димовски, Адитивни полугрупи на цели броеви, Prilozi IX, **2**, MANU, Skopje, 1977, p. 21-25, (**MR0554024** **Additive semigroups of integers**.(Macedonian) 10A99 (20M99));

[2] Д. Димовски, М. Хаџи-Коста Јосифовска: *Конечно генерирани потполугрупи од адитивната полугрупа* $\mathbb{N}^{n}$, Math. Maced. Vol1, (2003), 77-88 (**Finitely generated subsemigroups of the additive semigroup** $\mathbb{N}^{n}$); and

[3] М. Хаџи-Коста Јосифовска, Д. Димовски: *Опис на конечно генерирани адитивни подгрупи на* $\mathbb{Z}^{n}$, Збор. труд., III Конгрес на математичарите на Македонија, Струга 2005; (Сојуз на математичари на Македонија, Скопје 2007), 261-274 (**Description of additive subgroups of** $\mathbb{Z}^{n}$).

T.1.2. ([1]) Let G be a semigroup consisting of positive integers. Let n be the smallest integer in G, d the greatest common divisor of the elements of G and $n = kd$. Let us denote by A_i the set of those elements of G whose reminder after division by n is id, i.e.

$$A_i = \{a|a \in G, a = nt + id, t \in \mathbb{N}\}$$

(i) $G = A_0 \cup A_1 \cup \cdots \cup A_{k-1}$, the union is disjoint

(ii) There exist $1 = a_0, a_1, \dots, a_{k-1} \in \mathbb{N}$, such that $A_i = \{tn + id|\ t \geq a_i\}$ and

$$a_i + a_j \geq \begin{cases} a_{i+j}\ , i + j < k \\ a_{i+j-k} - 1\ , i + j \geq k \end{cases}$$

(iii) If $m_i = a_i n + id$ then $\{n = m_0, m_1, \dots, m_{k-1}\}$ generates G .

(iv) Let $b = max\{a_0, a_1, \dots, a_{k-1}\}$, $s = max\{i|a_i = b\}$ and $c = (b-1)k + s + 1$. Then $(c-1)d \notin G$ and $\{td|t \geq c\} = G_o \subseteq G$. (We say that G_o is the regular part of G).

T. 2.1. ([1])Let α be a congruence on *G* and $\alpha \neq \Delta_G$ (Δ_G is the diagonal). Then there exist $m, s_1, s_2, \ldots, s_{k-1} \in \mathbb{N}$ such that:

(i) $a\alpha b \Longrightarrow m|a - b$

(ii) For every $t \in \mathbb{N}_0$ $[(s_i + t)n + id]^\alpha$ is an infinite class, and for every $v \in A_i, v < s_i n + id \Longrightarrow v^\alpha$ is a finite class for $0 \leq i \leq k - 1$.

(iii) The integers s_i satisfy the following conditions: $s_i \geq a_i$ and

$$s_i + a_j \geq \begin{cases} s_{i+j}, i + j < k \\ s_{i+j-k} - 1, i + j \geq k \end{cases}$$

Examining the additive subsemigroups of $\mathbb{Z}^{\boldsymbol{n}}$ for n>1, first we come to the major diference with the case n=1. Any additive subsemigroup of $\mathbb{Z}$ is finitely generated which is not the case for n>1.

Thm.([2]) An additive subsemigroup G of $\mathbb{N}^{\boldsymbol{n}}$ for n>1 is finitely generated if and only if G is a subset of Cone(A) for some subset A of G.

In order to obtain better undesrstanding of the additive subsemigroups of $\mathbb{Z}^{\boldsymbol{n}}$ we needed a good description of the additive subgroups of $\mathbb{Z}^{\boldsymbol{n}}$, given in [3].

If in **T.1.2.** $d = 1$, then $G \cup \{0\}$ is a numerical semigroup whose multiplicity is *n*, conductor is *c*, gaps are all the numbers $tn + i$ for $t < a_i$, the genus is $\sum_{i=0}^{n-1} a_i$ and the Frobenius number is $c - 1$. We denote this semigroup by $G = [n; a_0 = 1, a_1, \dots, a_{n-1}]$. The notion of embedding dimension is not considered in [1].

Next, let $G = [n; a_0 = 1, a_1, \dots, a_{n-1}]$ be as above, let *R(G)* be the set of all $t \in \mathbb{Z}_n$, such that there are $i, j \in \mathbb{Z}_n$,$t = i \oplus j$ and $a_t = a_{i \oplus j} = a_i + a_j + [n; i, j]$ i.e.

$$R(G) = \left\{ i \oplus j | i, j \in \mathbb{Z}_n , a_{i \oplus j} = a_i + a_j + [n; i, j] \right\}$$

where $[n; i, j]$ is the integer part $\left[\frac{i+j}{n}\right]$, and let $S(G) = \mathbb{Z}_n \backslash R(G)$. We define two more sets:

$$B_0 = \{a_i n + i | i \in \mathbb{Z}_n\} \text{ and } M_0 = \{a_i n + i | i \in R(G)\} .$$

Theorem 1. The set $B_0 \backslash M_0$ is the minimal set of generators for *G*. So

$$ed(G) = |B_0 \backslash M_0| = |S(G)|.$$

For $n, i_1, i_2, \dots, i_k \in \mathbb{N}$ we denote the integer part $\left[\frac{i_1+i_2+\cdots i_k}{n}\right]$ by $[n; i_1, i_2, \dots, i_k]$

Fact. Let $n, k, t \in \mathbb{N}$ and $\{i_i, i_2, \dots, i_k\}, \{j_i, j_2, \dots, j_t\} \subseteq \mathbb{Z}_n$. *T*hen:

- $[n; i_1, i_2, \dots, i_k] = \frac{i_1+i_2+\cdots+i_k - i_1 \oplus i_2 \oplus \dots \oplus i_k}{n}$
- $[n; i_1, i_2, \dots, i_k, j_i, j_2, \dots, j_t] = [n; i_1 \oplus i_2 \oplus \dots \oplus i_k, j_i, j_2, \dots, j_t] + [n; i_1, i_2, \dots, i_k]$
- $[n; i_1, i_2, \dots, i_k, j_i, j_2, \dots, j_t] = [n; i_1 \oplus \dots \oplus i_k, j_1 \oplus \dots \oplus j_t] + [n; i_1, \dots, i_k] + [n; j_1, \dots, j_t]$
- $[n, i_1] = 0$

Fact. Let $G = [n; a_0 = 1, a_1, \dots, a_{n-1}]$. Then for each *k*,

$$a_{i_1 \oplus i_2 \oplus \dots \oplus i_k} \leq a_{i_1} + \cdots + a_{i_1} + [n; i_1, i_2, \dots, i_k]$$

Let n be a given positive integer. Let $T \subseteq \mathbb{Z}_n \backslash \{0\}$ be a generating set for $\mathbb{Z}_n$, and let $B(T) = \{b_s | s \in T\} \subseteq \mathbb{N}$ satisfies the following condition

$$\text{if } t \in T \text{and } t = i_1 \oplus \ldots \oplus i_r \text{ for } i_1, \ldots, i_r \in T \backslash \{t\}$$

$$\text{then } b_t < b_{i_1} + \ldots + b_{i_r} + [n; i_1, \ldots, i_r]. \quad \textbf{(*)}$$

We define a set $A = \{a_0, a_1, \ldots, a_{n-1}\}$ as follows:

(i) $a_0 = 1$,

(ii) If $i \in T$ then $a_i = b_i$,

(iii) If $i \notin T$ then

$$a_i = min\{ b_{i_1} + \ldots + b_{i_r} + [n; i_1, \ldots, i_r] \big| i = i_1 \oplus \ldots \oplus i_r, i_1, \ldots, i_r \in T\}.$$

Theorem. a) The numbers $a_0, a_1, \dots, a_{n-1}$ satisfy the condition (ii) of **T.1.2.**, and, so, $G = [n; a_0 = 1, a_1, \dots, a_{n-1}]$ =$[n; A]$ is a numerical semigroup. (We denote this semigruoup by $[n; T, B(T)]$)

b) $R([n; T; B(T)G]) = \mathbb{Z}_n \backslash (T \cup \{0\})$

c) $ed([n; T; B(T)]) = |T| + 1$

Theorem. A numerical semigroup G has $ed(G) = d$ if and only if $G = [n; T; B(T)]$ for some $n, T \subseteq \mathbb{Z}_n \backslash \{0\}$ and $B(T) \subseteq \mathbb{N}$ as above with $|T| = d - 1$.

Let $G = [n; T; B(T)], T = \{j_1, j_2, \dots, j_k\}, \mathcal{A} = \{na_s + s | s \in \mathbb{Z}_n\}$ and

$$\mathcal{M} = \{nb_{j_r} + j_r \big| r = 1,2, \dots, k\} = \{m_1, m_2, \dots, m_k\} .$$

We define $\varphi: \mathbb{Z}^k \longrightarrow \mathbb{Z}_n$ by:

$\varphi(z_1, z_2, \dots, z_k) = t$ iff $\sum_{s=1}^{k} z_s\, m_s$ is congruent to t modulo n.

Then, $\boldsymbol{H} = ker\varphi$ is an additive subgroup of of $\mathbb{Z}^k$ of rank k. Let

$\boldsymbol{H} \cap (\mathbb{N}_0)^k = \boldsymbol{B^0}; \boldsymbol{B} = \boldsymbol{B^0} \backslash \{0,0, \dots, 0\};\ \boldsymbol{D} = \boldsymbol{B} + (\mathbb{N}_0)^k$ and

$\boldsymbol{C} = (\mathbb{N}_0)^k \backslash \boldsymbol{D}$ (We say that $\boldsymbol{C}$ is the carier if G)

Theorem. For each $r \in \mathcal{A} \backslash \{0\},\ r = p_1 m_1 + \dots + p_k m_k$ for some $(p_1, \dots, p_{k)} \in \boldsymbol{C}$.

The group $\boldsymbol{H}$ is an invariant for the numerical semigroups.

$ed(G) = 2,\ G = [n; \{i\}, \{b_i\}],\ gcd(n,i)=1,\ x = b_i n + i,$
$\mathcal{M} = \{x\},\ \mathcal{A} = \{na_s + s | s \in \mathbb{Z}_n\} = \{m_s | s \in \mathbb{Z}_n\}$. The definition of G implies that

$$m_{t \odot i} = tx,$$

and so $F(G) = (n-1)x - n$.

$ed(G) = 3,\ G = [n; \{i, j\}, \{b_i, b_j\}]$, $\gcd(n, i) = \gcd(n, j) = 1$

$x = b_i n + i, y = b_j n + j$, $\mathcal{M} = \{x, y\}$, and

$\mathcal{A} = \{na_s + s | s \in \mathbb{Z}_n\} = \{m_s | s \in \mathbb{Z}_n\}$.

The definition of G implies that $m_s = \min\{px + qy | p \odot i \oplus q \odot j = s\}$.

If $p' \odot i = q' \odot j$ and $p'x > q'y$, then

$p' \odot i \oplus q \odot j = q' \odot j \oplus q \odot j = (q' + q) \odot j$ and $p'x + qy > (q' + q)y$.

So, we examine the minimal pairs $(p, -q) \in \boldsymbol{H},\ p, q \in \mathbb{N}$, where minimal means that there is no $(p', -q') \in \boldsymbol{H},\ p', q' \in \mathbb{N}$ such that $p' < p\ and\ q' < q$.

Fact: Let $(p,-q),(u,-v)$ be two such pairs satisfying: $p>u,\ \ q<v$ and

$$0<c<p, 0<d<v \implies (c,-d)\notin \boldsymbol{H} .$$

Then, simple calculation implies that $pv-qu=n$ and

$$\{s\odot i \oplus r\odot j \mid (s,r)\in A_L \cup A_R\}=\mathbb{Z}_n \text{ , where}$$

$$A_L=\{(s,r) \mid 0\le s<p, 0\le r<v-q\} \text{ and}$$

$$A_R=\{(s,r) \mid 0\le s<p-u, 0\le r<v\} .$$

Now, for $G = [n; \{i, j\}, \{b_i, b_j\}]$, $x = b_i n + i$, $y = b_j n + j$, let *p* be the smallest such that $px > (p \odot i \odot j^{-1})y$ and *v* be the smallest such that $vy > (v \odot j \odot i^{-1})y$.

Again, a simple calculation implies that the pairs

$$(p, -p \odot i \odot j^{-1}), (v \odot j \odot i^{-1}, -v)$$

satisfy the condition of the above Fact.

So, $\mathcal{A} = \{sx + ry | (s, r) \in A_L \cup A_R\}$ and

$F(G) = (p-1)x + (v-1)y - \min\{(v \odot j \odot i^{-1})x, (p \odot i \odot j^{-1})y\}$.

How do we find the minimal pairs $(p,-q) \in \boldsymbol{H}$?

We start with the minimal pairs $(n,0),(j,-1)$. The next minimal pair is:

$$\left(\left\lceil\frac{n}{j}\right\rceil j-n,-\left\lceil\frac{n}{j}\right\rceil\right).$$

If $(p,-q),(u,-v)$ are two consecutive minimal pairs, with $u \neq 0$, then the next minimal pair is

$$\left(\left\lceil\frac{p}{u}\right\rceil u-p,-\left\lceil\frac{p}{u}\right\rceil v-q\right).$$