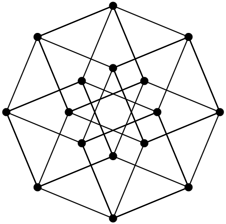
Congruences on ideals of semigroups and categories



James East

SandGAL

Politecnico di Milano, Cremona

12 June 2019

Joint work with Nik Ruškuc

Groups

Groups

Normal subgroups of the symmetric group $\mathcal{S}_n$

Groups

Normal subgroups of the symmetric group $\mathcal{S}_n$

There are three: $\{\mathrm{id}_n\} \trianglelefteq \mathcal{A}_n \trianglelefteq \mathcal{S}_n$.

Groups

Normal subgroups of the symmetric group $\mathcal{S}_n$

There are three: $\{\mathrm{id}_n\} \trianglelefteq \mathcal{A}_n \trianglelefteq \mathcal{S}_n$.

Exceptions:

- $\{\mathrm{id}_0\} = \mathcal{A}_0 = \mathcal{S}_0$,
- $\{\mathrm{id}_1\} = \mathcal{A}_1 = \mathcal{S}_1$,
- $\{\mathrm{id}_2\} = \mathcal{A}_2 \trianglelefteq \mathcal{S}_2$,

Groups

Normal subgroups of the symmetric group $\mathcal{S}_n$

There are three: $\{\mathrm{id}_n\} \trianglelefteq \mathcal{A}_n \trianglelefteq \mathcal{S}_n$.

Exceptions:

- $\{\mathrm{id}_0\} = \mathcal{A}_0 = \mathcal{S}_0$,
- $\{\mathrm{id}_1\} = \mathcal{A}_1 = \mathcal{S}_1$,
- $\{\mathrm{id}_2\} = \mathcal{A}_2 \trianglelefteq \mathcal{S}_2$,
- $\{\mathrm{id}_4\} \trianglelefteq K \trianglelefteq \mathcal{A}_4 \trianglelefteq \mathcal{S}_4$.

Groups

Normal subgroups of the symmetric group $\mathcal{S}_n$

There are three: $\{\mathrm{id}_n\} \trianglelefteq \mathcal{A}_n \trianglelefteq \mathcal{S}_n$.

Exceptions:

- $\{\mathrm{id}_0\} = \mathcal{A}_0 = \mathcal{S}_0$,
- $\{\mathrm{id}_1\} = \mathcal{A}_1 = \mathcal{S}_1$,
- $\{\mathrm{id}_2\} = \mathcal{A}_2 \trianglelefteq \mathcal{S}_2$,
- $\{\mathrm{id}_4\} \trianglelefteq K \trianglelefteq \mathcal{A}_4 \trianglelefteq \mathcal{S}_4$.

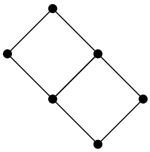
General shape of $\mathcal{N}(\mathcal{S}_n)$:

Groups

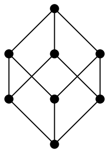
(Normal) subgroups of the cyclic group C_n

They correspond to the divisors of n.

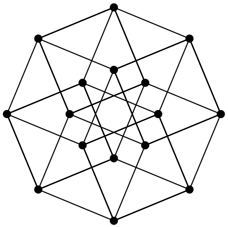
$\mathcal{N}(C_8)$



$\mathcal{N}(C_{12})$



$\mathcal{N}(C_{30})$

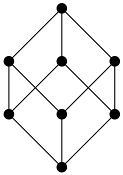


$\mathcal{N}(C_{210})$

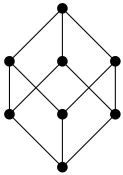
Groups

Normal subgroups of the dihedral group D_n

They correspond to the divisors of n (sort of); also depends on parity of n.



$\mathcal{N}(C_{30})$

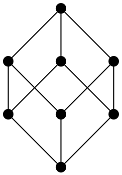


$\mathcal{N}(C_{105})$

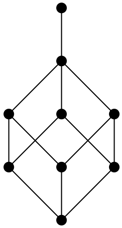
Groups

Normal subgroups of the dihedral group D_n

They correspond to the divisors of n (sort of); also depends on parity of n.



$\mathcal{N}(C_{30})$

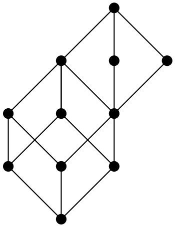


$\mathcal{N}(D_{105})$

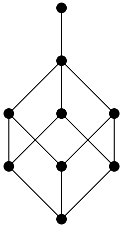
Groups

Normal subgroups of the dihedral group D_n

They correspond to the divisors of n (sort of); also depends on parity of n.



$\mathcal{N}(D_{30})$



$\mathcal{N}(D_{105})$

Congruences

Congruences

- Normal subgroups $\Leftrightarrow$ quotient groups.

Congruences

- Normal subgroups $\Leftrightarrow$ quotient groups.
- Ideals of rings $\Leftrightarrow$ quotient rings.

Congruences

- Normal subgroups $\Leftrightarrow$ quotient groups.
- Ideals of rings $\Leftrightarrow$ quotient rings.
- Groups and rings are lucky.

Congruences

- Normal subgroups $\Leftrightarrow$ quotient groups.
- Ideals of rings $\Leftrightarrow$ quotient rings.
- Groups and rings are lucky... Normally we need congruences.

Congruences

- Normal subgroups $\Leftrightarrow$ quotient groups.
- Ideals of rings $\Leftrightarrow$ quotient rings.
- Groups and rings are lucky... Normally we need congruences.

Definition (congruence on a structure S)

An equivalence on S compatible with its operation(s).

Congruences

- Normal subgroups $\Leftrightarrow$ quotient groups.
- Ideals of rings $\Leftrightarrow$ quotient rings.
- Groups and rings are lucky... Normally we need congruences.

Definition (congruence on a structure S)

An equivalence on S compatible with its operation(s).

Definition (congruence on a semigroup S)

An equivalence σ on S such that:

- $(x, y) \in \sigma \;\Rightarrow\; (ax, ay), (xa, ya) \in \sigma$ for all $a \in S$.

Congruences

- Normal subgroups $\Leftrightarrow$ quotient groups.
- Ideals of rings $\Leftrightarrow$ quotient rings.
- Groups and rings are lucky… Normally we need congruences.

Definition (congruence on a structure S)

An equivalence on S compatible with its operation(s).

Definition (congruence on a category S)

An equivalence σ on (morphisms of) S such that:

- $(x, y) \in \sigma \;\Rightarrow\; (ax, ay), (xa, ya) \in \sigma$ when products defined,
- $(x, y) \in \sigma \;\Rightarrow\; \mathbf{d}(x) = \mathbf{d}(y)$ and $\mathbf{r}(x) = \mathbf{r}(y)$.

Congruences

- Normal subgroups $\Leftrightarrow$ quotient groups.
- Ideals of rings $\Leftrightarrow$ quotient rings.
- Groups and rings are lucky... Normally we need congruences.

Definition (congruence on a structure S)

An equivalence on S compatible with its operation(s).

- The set $\mathrm{Cong}(S)$ of all congruences forms a lattice.

Congruences

- Normal subgroups $\Leftrightarrow$ quotient groups.
- Ideals of rings $\Leftrightarrow$ quotient rings.
- Groups and rings are lucky... Normally we need congruences.

Definition (congruence on a structure S)

An equivalence on S compatible with its operation(s).

- The set $\mathrm{Cong}(S)$ of all congruences forms a lattice.

Natural problem

Given S, find $\mathrm{Cong}(S)$.

Congruences

- Normal subgroups $\Leftrightarrow$ quotient groups.
- Ideals of rings $\Leftrightarrow$ quotient rings.
- Groups and rings are lucky... Normally we need congruences.

Definition (congruence on a structure S)

An equivalence on S compatible with its operation(s).

- The set $\mathrm{Cong}(S)$ of all congruences forms a lattice.

Natural problem

Given S, find $\mathrm{Cong}(S)$. Today S will usually be a semigroup.

Congruences on $\mathcal{T}_n$

Congruences on $\mathcal{T}_n$

- Let $\mathcal{T}_n$ = full transformation semigroup on $\mathbf{n} = \{1, \ldots, n\}$

  $= \{\text{functions } \mathbf{n} \to \mathbf{n}\}$.

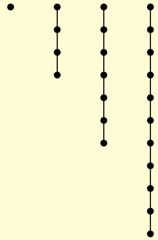
Congruences on $\mathcal{T}_n$

- Let $\mathcal{T}_n$ = full transformation semigroup on $\mathbf{n} = \{1, \ldots, n\}$

  $= \{\text{functions } \mathbf{n} \to \mathbf{n}\}$.

Theorem (Mal'cev, 1952)

n	1	2	3	4	5	6	7	8	9	10
Cong($\mathcal{T}_n$)										



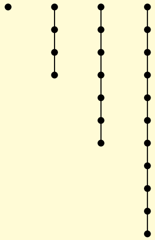
Congruences on $\mathcal{T}_n$

- Let $\mathcal{T}_n$ = full transformation semigroup on $\mathbf{n} = \{1, \ldots, n\}$

  $= \{\text{functions } \mathbf{n} \to \mathbf{n}\}$.

Theorem (Mal'cev, 1952)

n	1	2	3	4	5	6	7	8	9	10
Cong($\mathcal{T}_n$)										



- What are these congruences?

Ideals and Rees congruences

Ideals and Rees congruences

Ideal of a semigroup S

Ideals and Rees congruences

Ideal of a semigroup S

A nonempty subset I of S such that $IS \subseteq S$ and $SI \subseteq S$.

Ideals and Rees congruences

Ideal of a semigroup S

A nonempty subset I of S such that $IS \subseteq S$ and $SI \subseteq S$.

Rees congruence on a semigroup S

Ideals and Rees congruences

Ideal of a semigroup S

A nonempty subset I of S such that $IS \subseteq S$ and $SI \subseteq S$.

Rees congruence on a semigroup S

If I is an ideal of S, then we have a congruence:

$$R_I = \nabla_I \cup \Delta_S.$$

Ideals and Rees congruences

Ideal of a semigroup S

A nonempty subset I of S such that $IS \subseteq S$ and $SI \subseteq S$.

Rees congruence on a semigroup S

If I is an ideal of S, then we have a congruence:

$$R_I = \nabla_I \cup \Delta_S.$$

All of I is collapsed to a point.

Ideals and Rees congruences

Ideal of a semigroup S

A nonempty subset I of S such that $IS \subseteq S$ and $SI \subseteq S$.

Rees congruence on a semigroup S

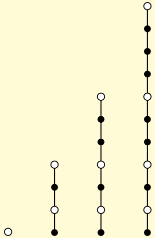
If I is an ideal of S, then we have a congruence:

$$R_I = \nabla_I \cup \Delta_S.$$

All of I is collapsed to a point. The rest of S is preserved.

Congruences on $\mathcal{T}_n$

Theorem (Mal'cev, 1952)

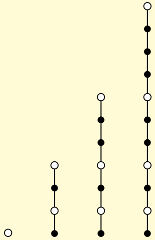
n	1	2	3	4	5	6	7	8	9	10
$\mathrm{Cong}(\mathcal{T}_n)$										

- Rees congruences are white.

Congruences on $\mathcal{T}_n$

Theorem (Mal'cev, 1952)

n	1	2	3	4	5	6	7	8	9	10
$\mathrm{Cong}(\mathcal{T}_n)$										



- Rees congruences are white.
- What are the other congruences?

Congruences on $\mathcal{T}_n$

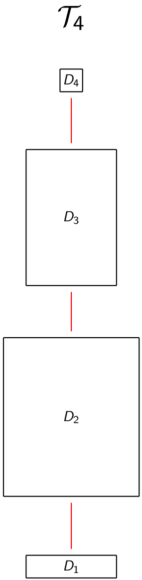
- For $\alpha \in \mathcal{T}_n$ define $\mathrm{rank}(\alpha) = |\mathrm{im}(\alpha)|$.

Congruences on $\mathcal{T}_n$

- For $\alpha \in \mathcal{T}_n$ define $\mathrm{rank}(\alpha) = |\mathrm{im}(\alpha)|$.
- For $1 \leq r \leq n$ define
  - $D_r = \{\alpha \in \mathcal{T}_n : \mathrm{rank}(\alpha) = r\}$,

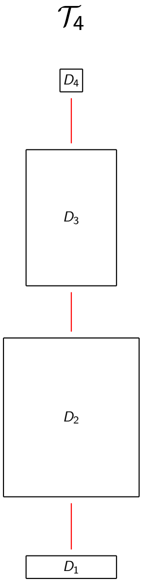
Congruences on $\mathcal{T}_n$

- For $\alpha \in \mathcal{T}_n$ define $\operatorname{rank}(\alpha) = |\operatorname{im}(\alpha)|$.
- For $1 \leq r \leq n$ define
  - $D_r = \{\alpha \in \mathcal{T}_n : \operatorname{rank}(\alpha) = r\}$,



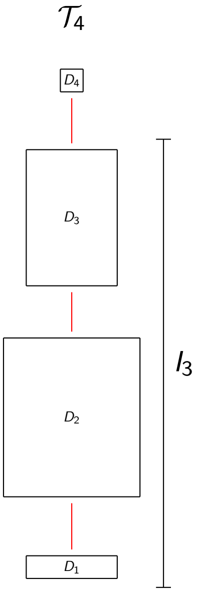
Congruences on $\mathcal{T}_n$

- For $\alpha \in \mathcal{T}_n$ define $\mathrm{rank}(\alpha) = |\mathrm{im}(\alpha)|$.
- For $1 \leq r \leq n$ define
  - $D_r = \{\alpha \in \mathcal{T}_n : \mathrm{rank}(\alpha) = r\}$,
  - $I_r = \{\alpha \in \mathcal{T}_n : \mathrm{rank}(\alpha) \leq r\}$.



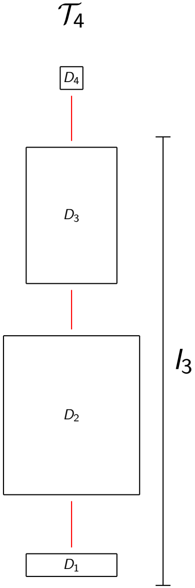
Congruences on $\mathcal{T}_n$

- For $\alpha \in \mathcal{T}_n$ define $\operatorname{rank}(\alpha) = |\operatorname{im}(\alpha)|$.
- For $1 \leq r \leq n$ define
  - $D_r = \{\alpha \in \mathcal{T}_n : \operatorname{rank}(\alpha) = r\}$,
  - $I_r = \{\alpha \in \mathcal{T}_n : \operatorname{rank}(\alpha) \leq r\}$.



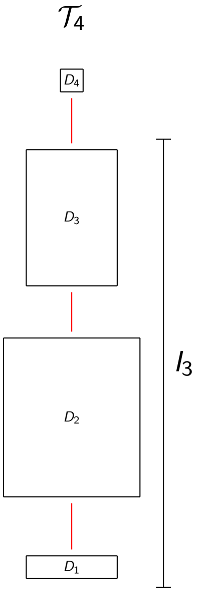
Congruences on $\mathcal{T}_n$

- For $\alpha \in \mathcal{T}_n$ define $\mathrm{rank}(\alpha) = |\mathrm{im}(\alpha)|$.
- For $1 \le r \le n$ define
  - $D_r = \{\alpha \in \mathcal{T}_n : \mathrm{rank}(\alpha) = r\}$,
  - $I_r = \{\alpha \in \mathcal{T}_n : \mathrm{rank}(\alpha) \le r\}$.
- Ideals: $I_1 \subset I_2 \subset \cdots \subset I_n = \mathcal{T}_n$.



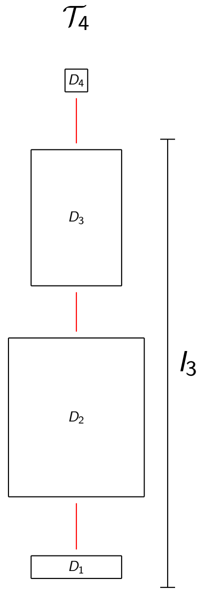
Congruences on $\mathcal{T}_n$

- For $\alpha \in \mathcal{T}_n$ define $\text{rank}(\alpha) = |\text{im}(\alpha)|$.
- For $1 \leq r \leq n$ define
  - $D_r = \{\alpha \in \mathcal{T}_n : \text{rank}(\alpha) = r\}$,
  - $I_r = \{\alpha \in \mathcal{T}_n : \text{rank}(\alpha) \leq r\}$.
- Ideals: $I_1 \subset I_2 \subset \cdots \subset I_n = \mathcal{T}_n$.
- Rees congruences: $R_{I_1} \subset R_{I_2} \subset \cdots \subset R_{I_n} = \nabla$.



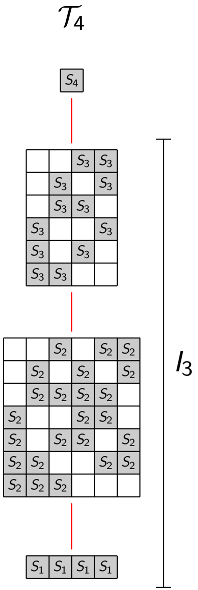
Congruences on $\mathcal{T}_n$

- For $\alpha \in \mathcal{T}_n$ define $\text{rank}(\alpha) = |\text{im}(\alpha)|$.
- For $1 \leq r \leq n$ define
  - $D_r = \{\alpha \in \mathcal{T}_n : \text{rank}(\alpha) = r\}$,
  - $I_r = \{\alpha \in \mathcal{T}_n : \text{rank}(\alpha) \leq r\}$.
- Ideals: $I_1 \subset I_2 \subset \cdots \subset I_n = \mathcal{T}_n$.
- Rees congruences: $R_{I_1} \subset R_{I_2} \subset \cdots \subset R_{I_n} = \nabla$.
- Inside D_r are lots of little groups $\cong \mathcal{S}_r$.



Congruences on $\mathcal{T}_n$

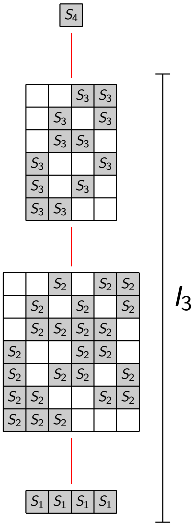
- For $\alpha \in \mathcal{T}_n$ define $\mathrm{rank}(\alpha) = |\mathrm{im}(\alpha)|$.
- For $1 \leq r \leq n$ define
  - $D_r = \{\alpha \in \mathcal{T}_n : \mathrm{rank}(\alpha) = r\}$,
  - $I_r = \{\alpha \in \mathcal{T}_n : \mathrm{rank}(\alpha) \leq r\}$.
- Ideals: $I_1 \subset I_2 \subset \cdots \subset I_n = \mathcal{T}_n$.
- Rees congruences: $R_{I_1} \subset R_{I_2} \subset \cdots \subset R_{I_n} = \nabla$.
- Inside D_r are lots of little groups $\cong \mathcal{S}_r$.



Congruences on $\mathcal{T}_n$

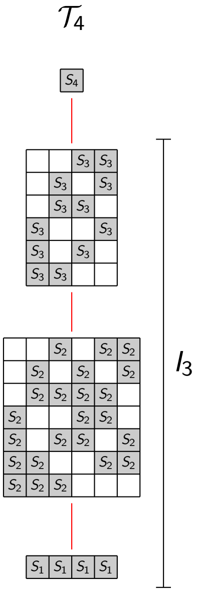
- For $\alpha \in \mathcal{T}_n$ define $\text{rank}(\alpha) = |\text{im}(\alpha)|$.
- For $1 \leq r \leq n$ define
  - $D_r = \{\alpha \in \mathcal{T}_n : \text{rank}(\alpha) = r\}$,
  - $I_r = \{\alpha \in \mathcal{T}_n : \text{rank}(\alpha) \leq r\}$.
- Ideals: $I_1 \subset I_2 \subset \cdots \subset I_n = \mathcal{T}_n$.
- Rees congruences: $R_{I_1} \subset R_{I_2} \subset \cdots \subset R_{I_n} = \nabla$.
- Inside D_r are lots of little groups $\cong \mathcal{S}_r$.
- Each $N \trianglelefteq \mathcal{S}_r$ gives another congruence R_N:

$\mathcal{T}_4$



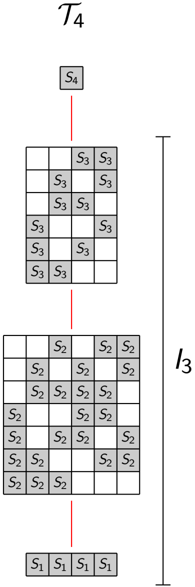
Congruences on $\mathcal{T}_n$

- For $\alpha \in \mathcal{T}_n$ define $\operatorname{rank}(\alpha) = |\operatorname{im}(\alpha)|$.
- For $1 \le r \le n$ define
  - $D_r = \{\alpha \in \mathcal{T}_n : \operatorname{rank}(\alpha) = r\}$,
  - $I_r = \{\alpha \in \mathcal{T}_n : \operatorname{rank}(\alpha) \le r\}$.
- Ideals: $I_1 \subset I_2 \subset \cdots \subset I_n = \mathcal{T}_n$.
- Rees congruences: $R_{I_1} \subset R_{I_2} \subset \cdots \subset R_{I_n} = \nabla$.
- Inside D_r are lots of little groups $\cong \mathcal{S}_r$.
- Each $N \trianglelefteq \mathcal{S}_r$ gives another congruence R_N:
  - each $\mathcal{S}_r$ collapses to $\mathcal{S}_r/N$,



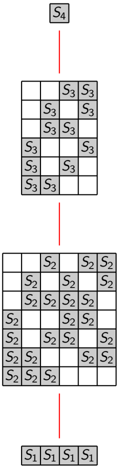
Congruences on $\mathcal{T}_n$

- For $\alpha \in \mathcal{T}_n$ define $\text{rank}(\alpha) = |\text{im}(\alpha)|$.
- For $1 \leq r \leq n$ define
  - $D_r = \{\alpha \in \mathcal{T}_n : \text{rank}(\alpha) = r\}$,
  - $I_r = \{\alpha \in \mathcal{T}_n : \text{rank}(\alpha) \leq r\}$.
- Ideals: $I_1 \subset I_2 \subset \cdots \subset I_n = \mathcal{T}_n$.
- Rees congruences: $R_{I_1} \subset R_{I_2} \subset \cdots \subset R_{I_n} = \nabla$.
- Inside D_r are lots of little groups $\cong \mathcal{S}_r$.
- Each $N \trianglelefteq \mathcal{S}_r$ gives another congruence R_N:
  - each $\mathcal{S}_r$ collapses to $\mathcal{S}_r/N$,
  - all of I_{r-1} collapses to a point.

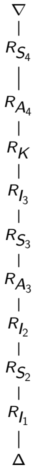


Congruences on $\mathcal{T}_n$

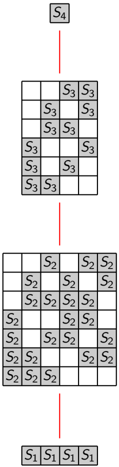
$\mathcal{T}_4$



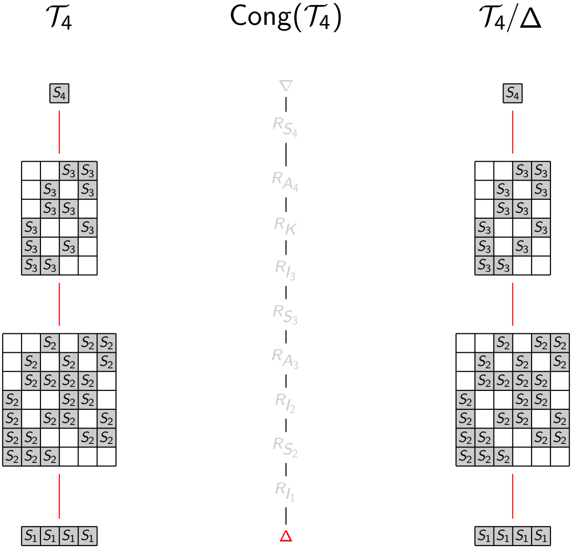
Cong($\mathcal{T}_4$)



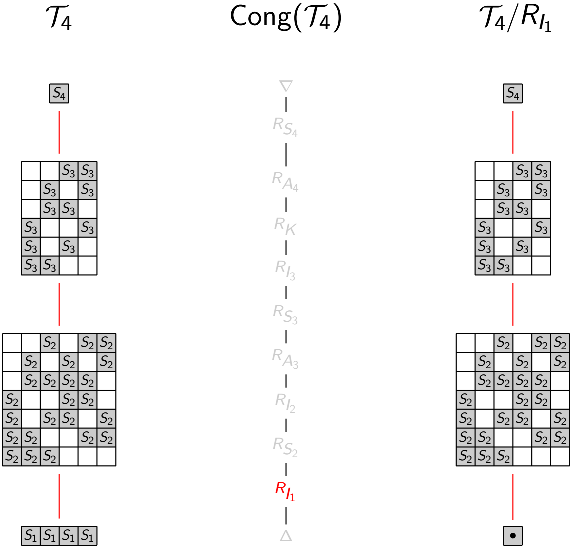
$\mathcal{T}_4$



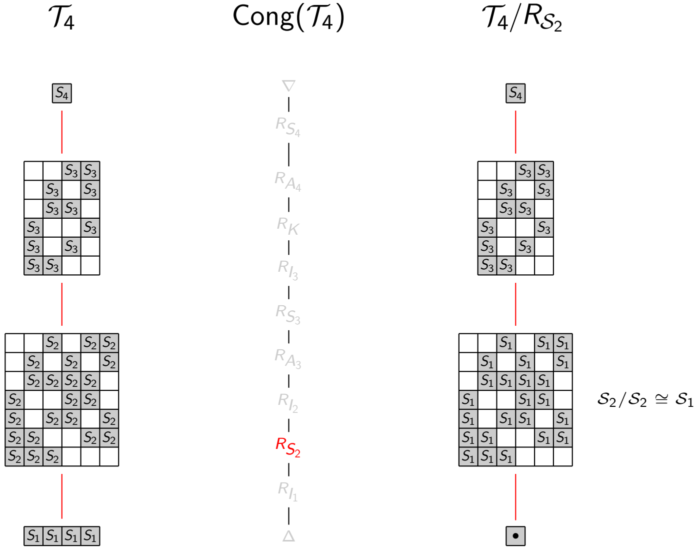
Congruences on $\mathcal{T}_n$



Congruences on $\mathcal{T}_n$

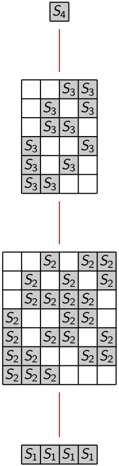


Congruences on $\mathcal{T}_n$

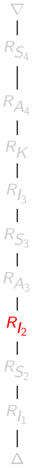


Congruences on $\mathcal{T}_n$

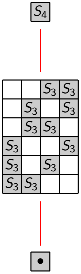
$\mathcal{T}_4$



$\mathrm{Cong}(\mathcal{T}_4)$

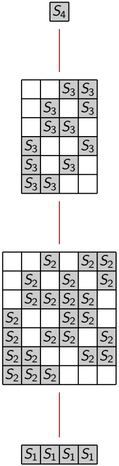


$\mathcal{T}_4/R_{I_2}$



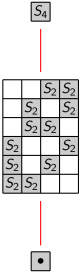
Congruences on $\mathcal{T}_n$

$\mathcal{T}_4$



$\mathrm{Cong}(\mathcal{T}_4)$

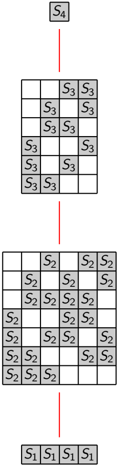
$\mathcal{T}_4/R_{\mathcal{A}_3}$



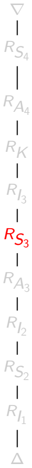
$\mathcal{S}_3/\mathcal{A}_3 \cong \mathcal{S}_2$

Congruences on $\mathcal{T}_n$

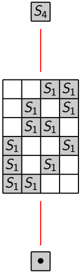
$\mathcal{T}_4$



$\mathrm{Cong}(\mathcal{T}_4)$



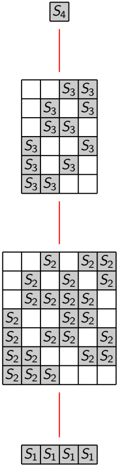
$\mathcal{T}_4/R_{\mathcal{S}_3}$

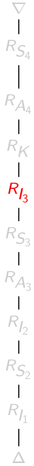


$\mathcal{S}_3/\mathcal{S}_3 \cong \mathcal{S}_1$

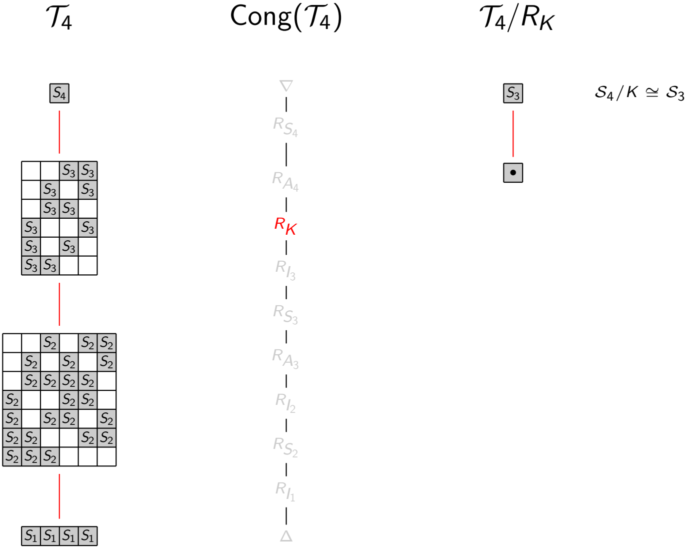
Congruences on $\mathcal{T}_n$

$\mathcal{T}_4$ | $\mathrm{Cong}(\mathcal{T}_4)$ | $\mathcal{T}_4/R_{I_3}$



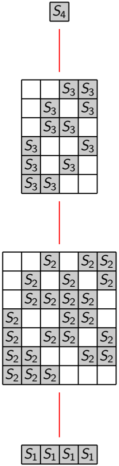


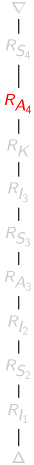
Congruences on $\mathcal{T}_n$



Congruences on $\mathcal{T}_n$

$\mathcal{T}_4$ | $\mathrm{Cong}(\mathcal{T}_4)$ | $\mathcal{T}_4/R_{\mathcal{A}_4}$

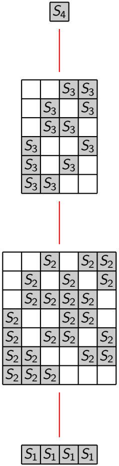




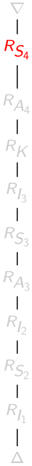
$\mathcal{S}_4/\mathcal{A}_4 \cong \mathcal{S}_2$

Congruences on $\mathcal{T}_n$

$\mathcal{T}_4$



$\mathrm{Cong}(\mathcal{T}_4)$

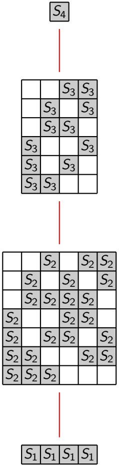


$\mathcal{T}_4/R_{\mathcal{S}_4}$

$\mathcal{S}_4/\mathcal{S}_4 \cong \mathcal{S}_1$

Congruences on $\mathcal{T}_n$

$\mathcal{T}_4$ | $\mathrm{Cong}(\mathcal{T}_4)$ | $\mathcal{T}_4/\nabla$



Congruences on ideals

Congruences on ideals

- An ideal I of S leads to a congruence R_I on S.

Congruences on ideals

- An ideal I of S leads to a congruence R_I on S.
- But I is also a semigroup!

Congruences on ideals

- An ideal I of S leads to a congruence R_I on S.
- But I is also a semigroup!
- What are the congruences on I?

Congruences on ideals

- An ideal I of S leads to a congruence R_I on S.
- But I is also a semigroup!
- What are the congruences on I?

Natural problem

Given S, find $\mathrm{Cong}(I)$ for each ideal I of S.

Congruences on ideals

- An ideal I of S leads to a congruence R_I on S.
- But I is also a semigroup!
- What are the congruences on I?

Natural problem

Given S, find $\mathrm{Cong}(I)$ for each ideal I of S.

Natural problem

Can we describe $\mathrm{Cong}(I_r)$, where $I_r = I_r(\mathcal{T}_n)$?

Congruences on ideals

- An ideal I of S leads to a congruence R_I on S.
- But I is also a semigroup!
- What are the congruences on I?

Natural problem

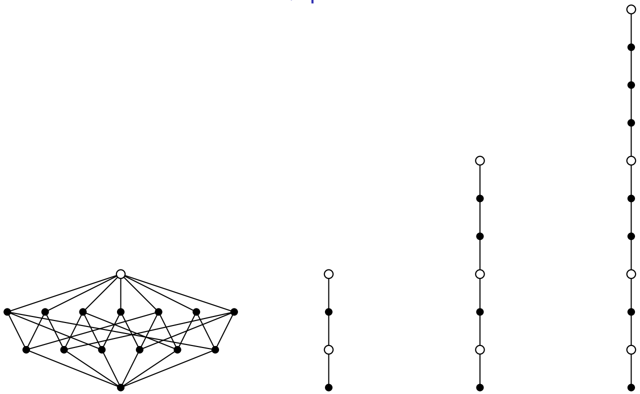
Given S, find $\mathrm{Cong}(I)$ for each ideal I of S.

Natural problem

Can we describe $\mathrm{Cong}(I_r)$, where $I_r = I_r(\mathcal{T}_n)$?

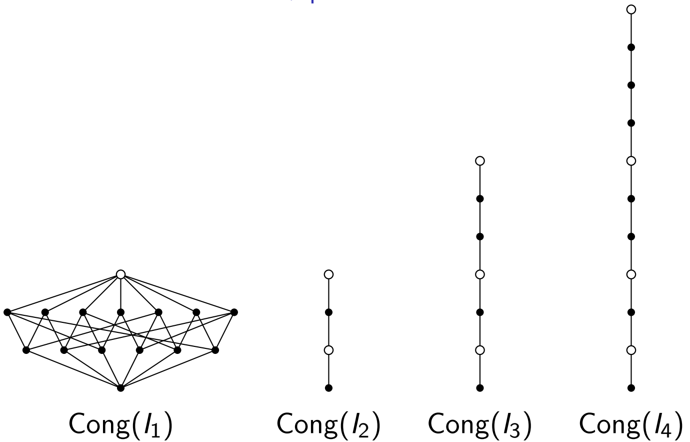
- Let's ask GAP!

Congruences on ideals of $\mathcal{T}_4$



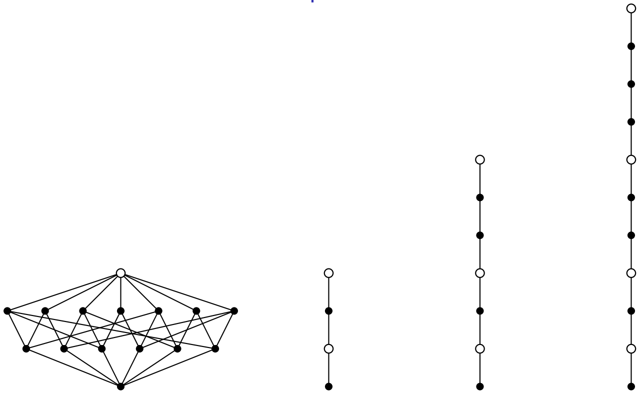
$\mathrm{Cong}(I_1)$ $\mathrm{Cong}(I_2)$ $\mathrm{Cong}(I_3)$ $\mathrm{Cong}(I_4)$

Congruences on ideals of $\mathcal{T}_4$



- I_1 is an n-element right-zero semigroup: $\text{Cong}(I_1) \cong \mathfrak{Eq}_n$.

Congruences on ideals of $\mathcal{T}_4$



- I_1 is an n-element right-zero semigroup: $\mathrm{Cong}(I_1) \cong \mathfrak{Eq}_n$.
- For $r \geq 2$, is $\mathrm{Cong}(I_r)$ just $\mathrm{Cong}(\mathcal{T}_n)$ chopped off?

Congruences on ideals of $\mathcal{T}_n$

Theorem

Yes!

Congruences on ideals of $\mathcal{T}_n$

Theorem

- $\mathrm{Cong}(I_1) \cong \mathfrak{Eq}_n$.
- $\mathrm{Cong}(I_r) = \left\{ R_N^{I_r} : N \trianglelefteq \mathcal{S}_q,\ q \leq n \right\} \cup \left\{ \nabla_{I_r} \right\}$ for $2 \leq r \leq n$.

Congruences on ideals of $\mathcal{T}_n$

Theorem

- $\mathrm{Cong}(I_1) \cong \mathfrak{Eq}_n$.
- $\mathrm{Cong}(I_r) = \{R_N^{I_r} : N \trianglelefteq \mathcal{S}_q,\ q \leq n\} \cup \{\nabla_{I_r}\}$ for $2 \leq r \leq n$.

- Original proof strategy:
  - Deal with I_1 and I_r $(r \geq 2)$ separately.
  - Use knowledge about $\mathrm{Cong}(\mathcal{T}_n)$.

Congruences on ideals of $\mathcal{T}_n$

Theorem

- $\mathrm{Cong}(I_1) \cong \mathfrak{Eq}_n$.
- $\mathrm{Cong}(I_r) = \{R_N^{I_r} : N \trianglelefteq \mathcal{S}_q,\ q \leq n\} \cup \{\nabla_{I_r}\}$ for $2 \leq r \leq n$.

- Original proof strategy:
  - Deal with I_1 and I_r $(r \geq 2)$ separately.
  - Use knowledge about $\mathrm{Cong}(\mathcal{T}_n)$.
- Later: general machinery that works for many other semigroups and categories...
  - transformations, linear transformations, diagrams, braids...

Congruences on ideals of $\mathcal{T}_n$

Theorem

- $\mathrm{Cong}(I_1) \cong \mathfrak{Eq}_n$.
- $\mathrm{Cong}(I_r) = \{R_N^{I_r} : N \trianglelefteq \mathcal{S}_q,\ q \leq n\} \cup \{\nabla_{I_r}\}$ for $2 \leq r \leq n$.

- Original proof strategy:
  - Deal with I_1 and I_r ($r \geq 2$) separately.
  - Use knowledge about $\mathrm{Cong}(\mathcal{T}_n)$.
- Later: general machinery that works for many other semigroups and categories...
  - transformations, linear transformations, diagrams, braids...
  - Treat smallest ideal(s) of S, then "lift" from one to the next.

Congruences on ideals of $\mathcal{T}_n$

Theorem

- $\mathrm{Cong}(I_1) \cong \mathfrak{Eq}_n$.
- $\mathrm{Cong}(I_r) = \{R_N^{I_r} : N \trianglelefteq \mathcal{S}_q,\ q \leq n\} \cup \{\nabla_{I_r}\}$ for $2 \leq r \leq n$.

- Original proof strategy:
  - Deal with I_1 and I_r $(r \geq 2)$ separately.
  - Use knowledge about $\mathrm{Cong}(\mathcal{T}_n)$.
- Later: general machinery that works for many other semigroups and categories...
  - transformations, linear transformations, diagrams, braids...
  - Treat smallest ideal(s) of S, then "lift" from one to the next.
  - No need to know $\mathrm{Cong}(S)$ in advance.

Congruences on ideal extensions

Congruences on ideal extensions

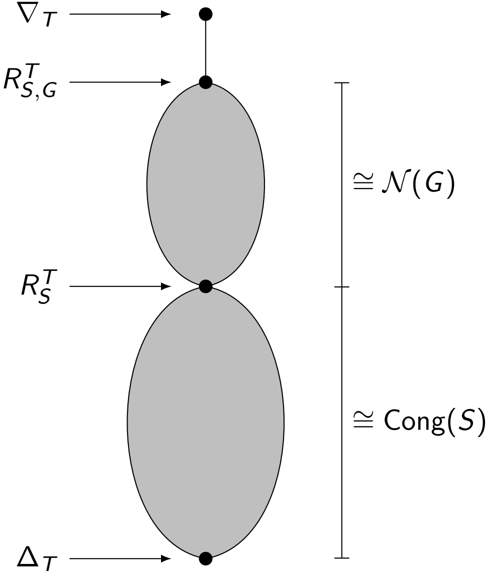
Theorem

- Suppose T is a semigroup with a stable, regular maximum $\mathscr{J}$-class D_T.
- Suppose the ideal $S = T \setminus D_T$ has a stable, regular maximum $\mathscr{J}$-class D_S.
- Suppose $(x, y)^\sharp = \nabla_S$ for all $x \in D_S$ and $y \in S \setminus H_x$.
- Suppose every congruence on S is liftable to T.
- One more technical assumption.
- Let G be a group $\mathscr{H}$-class contained in D_T.

Then

$$\operatorname{Cong}(T) = \left\{\Delta_{D_T} \cup \sigma : \sigma \in \operatorname{Cong}(S)\right\} \cup \left\{R_{S,N}^T : N \trianglelefteq G\right\} \cup \left\{\nabla_T\right\}.$$

Congruences on ideal extensions



Congruences on ideal extensions

Congruences on ideal extensions

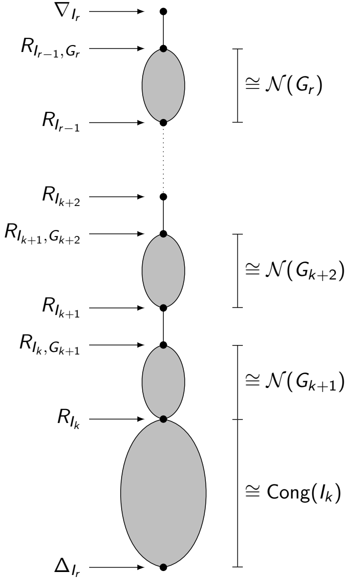
Theorem

- Let S be a stable, regular partial semigroup with a chain of $\mathscr{J}$-classes $D_0 < D_1 < \cdots$.
- The ideals of S are $I_r = D_0 \cup \cdots \cup D_r$ (and $I_\omega = S$ if the chain is infinite).
- Let G_q be a group $\mathscr{H}$-class in D_q.
- Suppose for some k every congruence on I_k is liftable to S.
- A technical property on I_k, and another on $I_{k+1}, I_{k+2}, \ldots$

Then for any $r \geq k$ (including $r = \omega$),

$$\begin{aligned}\mathrm{Cong}(I_r) = &\left\{\Delta_{I_r} \cup \sigma : \sigma \in \mathrm{Cong}(I_k)\right\} \\ &\cup \left\{R^{I_r}_{I_q,N} : k \leq q < r,\ N \trianglelefteq G_{q+1}\right\} \cup \left\{\nabla_{I_r}\right\}.\end{aligned}$$

Congruences on ideal extensions



More applications

More applications

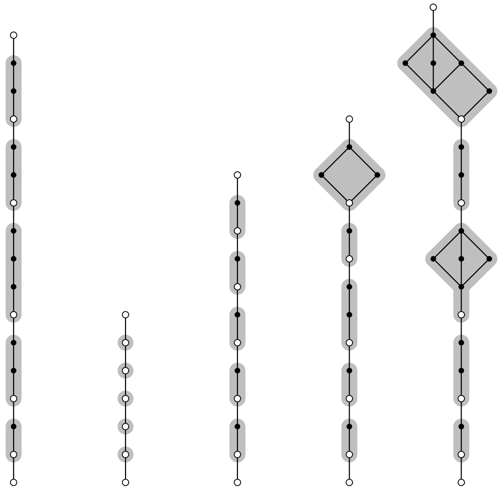
- Full transformation categories $\mathcal{T} = \mathcal{T}(\mathscr{C})$

More applications

- Full transformation categories $\mathcal{T} = \mathcal{T}(\mathscr{C})$
  - Subcategories preserving/reversing order/orientation: $\mathcal{O}$, $\mathcal{OD}$, $\mathcal{OP}$, $\mathcal{OR}$.

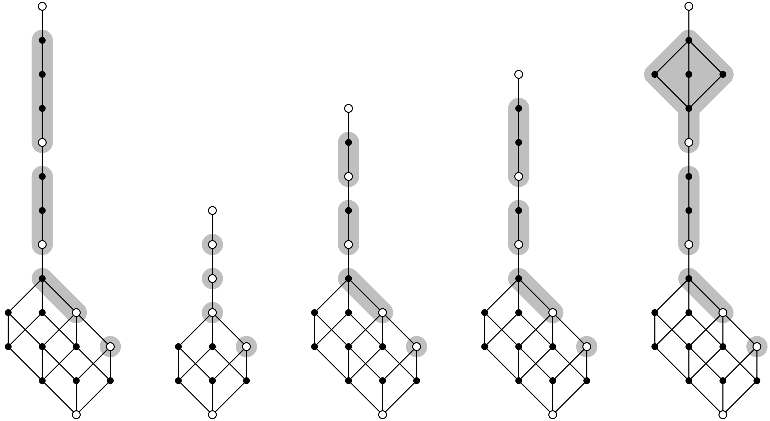
More applications

- Full transformation categories $\mathcal{T} = \mathcal{T}(\mathscr{C})$
  - Subcategories preserving/reversing order/orientation: $\mathcal{O}$, $\mathcal{OD}$, $\mathcal{OP}$, $\mathcal{OR}$.



More applications

- Partition categories $\mathcal{P} = \mathcal{P}(\mathscr{C})$
  - Planar, anti-planar, annular, anti-annular subcategories.



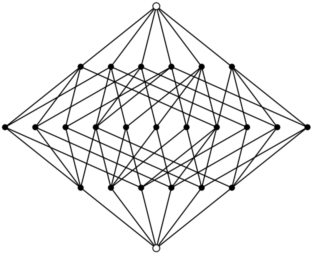
$\mathcal{P}_4$ $\mathscr{P}(\mathcal{P}_4)$ $\mathscr{P}^{\pm}(\mathcal{P}_4)$ $\mathscr{A}(\mathcal{P}_4)$ $\mathscr{A}^{\pm}(\mathcal{P}_4)$

More applications

- Brauer categories $\mathcal{B}$

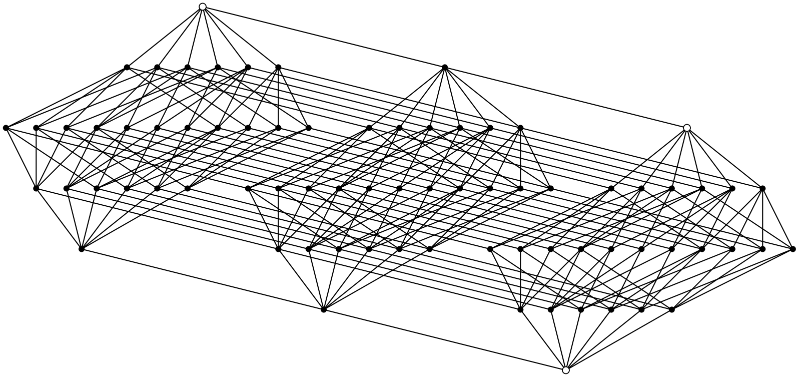
More applications

- Brauer categories $\mathcal{B}$
- $I_0(\mathcal{B}_4)$: $\quad \mathfrak{Eq}_3 \times \mathfrak{Eq}_3$



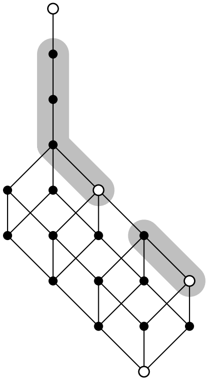
More applications

- Brauer categories $\mathcal{B}$
- $I_2(\mathcal{B}_4)$: $\quad (\mathfrak{Eq}_3 \times \mathfrak{Eq}_3) \times \mathbf{3}$



More applications

- Brauer categories $\mathcal{B}$
- $I_4(\mathcal{B}_4)$

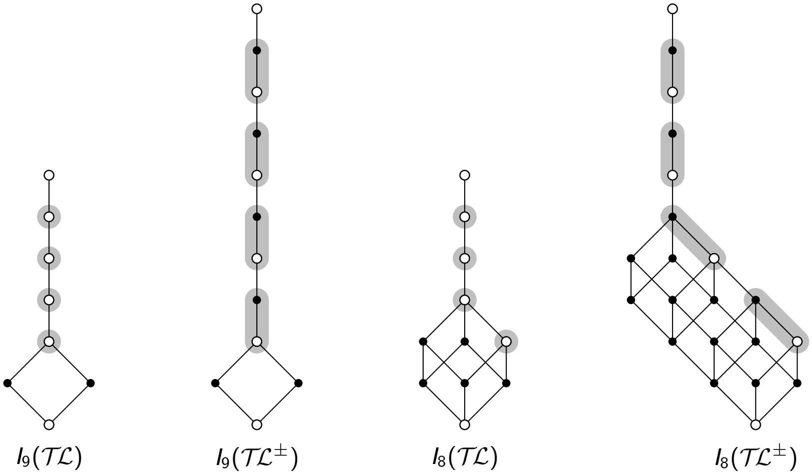


More applications

- Brauer categories $\mathcal{B}$
- (Anti-)planar/annular subcategories: Temperley-Lieb, Jones...

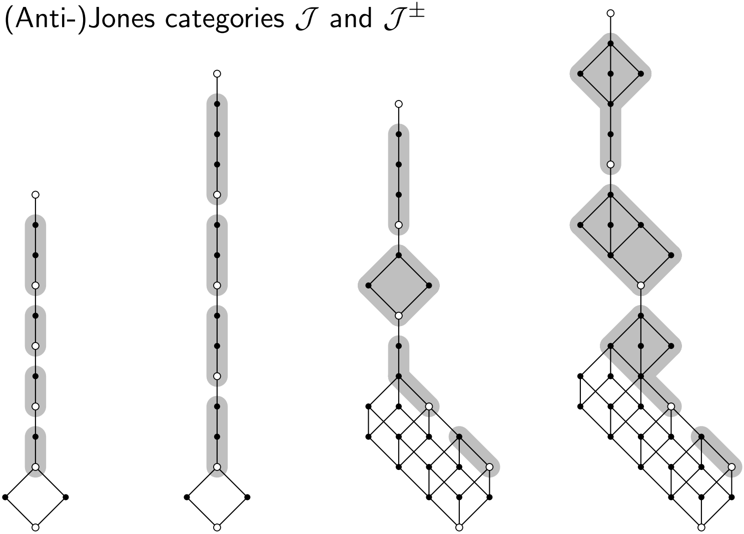
More applications

- (Anti-)Temperley-Lieb categories $\mathcal{TL}$ and $\mathcal{TL}^{\pm}$



More applications

- (Anti-)Jones categories $\mathcal{J}$ and $\mathcal{J}^{\pm}$



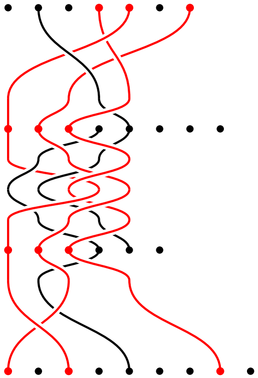
$I_9(\mathcal{J})$ $I_9(\mathcal{J}^{\pm})$ $I_8(\mathcal{J})$ $I_8(\mathcal{J}^{\pm})$

More applications

- Some semigroups/categories with chains of ideals don't fit the mould of the above theorems.

More applications

- Some semigroups/categories with chains of ideals don't fit the mould of the above theorems.
- Examples include linear categories and partial braid categories.



More applications

- Some semigroups/categories with chains of ideals don't fit the mould of the above theorems.
- Examples include linear categories and partial braid categories.
- These have nontrivial congruences contained in $\mathscr{H}$.

More applications

- Some semigroups/categories with chains of ideals don't fit the mould of the above theorems.
- Examples include linear categories and partial braid categories.
- These have nontrivial congruences contained in $\mathscr{H}$.
- We have general results to deal with (some of) these.

More applications

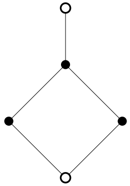
- Some semigroups/categories with chains of ideals don't fit the mould of the above theorems.
- Examples include linear categories and partial braid categories.
- These have nontrivial congruences contained in $\mathscr{H}$.
- We have general results to deal with (some of) these.
- Congruences of form $R^{I_r}_{I_q, N_{q+1}, N_{q+2}, \ldots}$, with $N_{q+1} \succeq N_{q+2} \succeq \cdots$

More applications

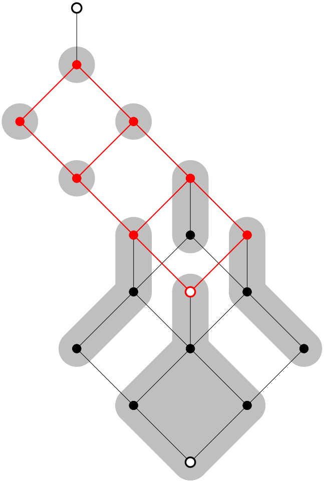
- Some semigroups/categories with chains of ideals don't fit the mould of the above theorems.
- Examples include linear categories and partial braid categories.
- These have nontrivial congruences contained in $\mathscr{H}$.
- We have general results to deal with (some of) these.
- Congruences of form $R^{I_r}_{I_q, N_{q+1}, N_{q+2}, \ldots}$, with $N_{q+1} \succeq N_{q+2} \succeq \cdots$
- Can still build $\mathrm{Cong}(I_{r+1})$ from $\mathrm{Cong}(I_r)$.
  - It's just more complicated...

More applications

- Linear category $\mathcal{L} = \mathcal{L}(\mathbb{F}_7)$



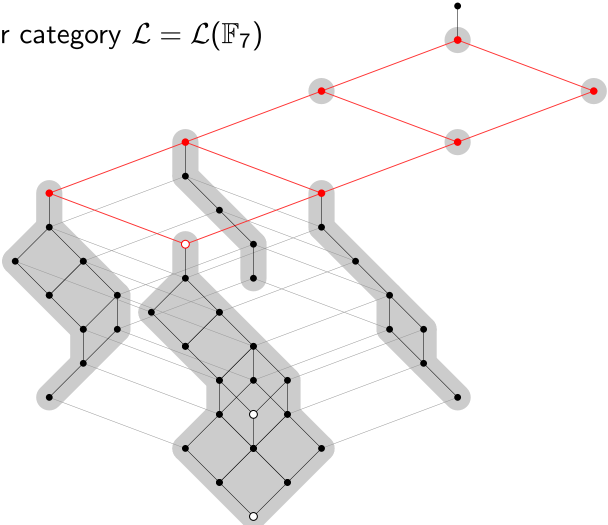
$I_1(\mathcal{L})$



$I_2(\mathcal{L})$

More applications

- Linear category $\mathcal{L} = \mathcal{L}(\mathbb{F}_7)$



$I_3(\mathcal{L})$

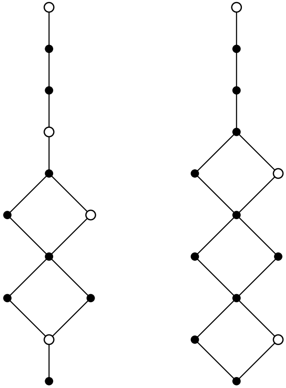
Current/future work

Current/future work

- Other categories:

Current/future work

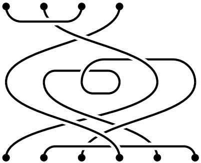
- Other categories:
  - twisted diagram categories

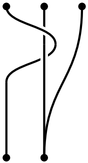


$\mathcal{P}_3^0$ $\mathcal{B}_4^0$

Current/future work

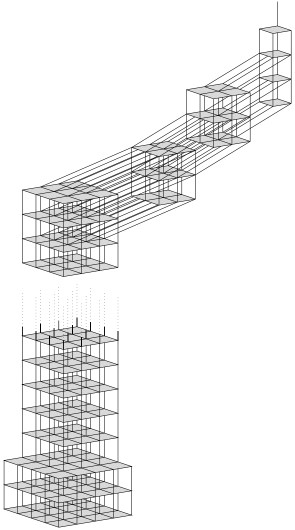
- Other categories:
  - twisted diagram categories
  - tangle/vine categories





Current/future work

- Other categories:
  - twisted diagram categories
  - tangle/vine categories
  - transformations/diagrams with infinite underlying sets



Current/future work

- Other categories:
  - twisted diagram categories
  - tangle/vine categories
  - transformations/diagrams
    with infinite underlying sets
- Some fit our general framework, some don't

Current/future work

- Other categories:
  - twisted diagram categories
  - tangle/vine categories
  - transformations/diagrams
    with infinite underlying sets
- Some fit our general framework, some don't
- One-sided ideals

Current/future work

- Other categories:
  - twisted diagram categories
  - tangle/vine categories
  - transformations/diagrams
    with infinite underlying sets
- Some fit our general framework, some don't
- One-sided ideals
- Variants/sandwich semigroups

Grazie mille :-)

Congruences lattices of ideals in categories and (partial) semigroups

- James East and Nik Ruškuc
- Coming soon to arXiv...