AFFINE STEINER LOOPS

Giovanni Falcone, Palermo

(joint work with Agota Figula and Carolin Hannusch, Debrecen)

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Steiner triple system: a pair $S = (P, T)$ where

P is a set of elements, called **points**

T is a family of **triples** $\{a, b, c\}$ of points of P, such that

ANY TWO POINTS IN P BELONG EXACTLY TO ONE TRIPLE IN T

Basic example 1: lines in an affine geometry AG($d, 3$) over GF(3)

Basic example 2: lines in a projective geometry PG($d, 2$) over GF(2)

STS have a long history, going back to Kirkman's schoolgirl problem (1846)

The term Steiner loop usually refers to the following structure:

i) for a triple $\{a, b, c\}$ in T, define $a + b = c$

ii) add an outer zero element $\Omega \notin P$

iii) define $a + a = \Omega$

Note that, doing this for PG$(d, 2)$ one gets $\left(\mathbb{Z}/2\mathbb{Z}\right)^{d+1}$

Note that in PG($d,2$)

$$a + b + c = \Omega$$

... and in AG($d,3$), as well !

A. Caggegi, GF, M. Pavone

On the additivity of block designs

J. Alg. Comb. 2017

The only STS which can be embedded in a group such that

$$a + b + c = 0$$

are PG(d, 2) and AG(d, 3)

Projective Steiner loops vs *Affine* Steiner loops

i) choose a zero element $\Omega \in P$

ii) define $-a$ through the triple $\{a, \Omega, -a\}$

iii) define $a + b = -c$ through the triple $\{a, b, c\}$

iv) define $a + a + a = \Omega$

Hall triple systems: any two triples generate an AG$(2,3)$

$$x + y = -(-x - y)$$

Let S be a Steiner triple system and L_S the corresponding affine Steiner loop.

i) L_R is a subloop of L_S if and only if R is a Steiner triple subsystem of S containing Ω.

ii) If L_R is a normal subloop of L_S then each coset $x + L_R$ corresponds to a subsystem of S.

iii) If L_R is a normal subloop of L_S then the quotient loop Q yields a Steiner triple system S_Q.

It must be noticed that, in the case where L_R is a normal subloop of L_S, the corresponding cosets are not necessarily isomorphic subsystems

S a Steiner triple system

L_S, Ω the corresponding affine Steiner loop.

$$\mathrm{Aut}(S)_\Omega = \mathrm{Aut}(L_S).$$

Any affine Steiner loop $\mathcal{L}_S$ has a subnormal series

$$\Omega \trianglelefteq \mathcal{L}_{S_1} \trianglelefteq \cdots \trianglelefteq \mathcal{L}_{S_t} = \mathcal{L}_S,$$

where the factors $\mathcal{L}_{S_{i+1}}/\mathcal{L}_{S_i}$ are simple affine Steiner loops.

If $\mathcal{L}_S$ is a HTS, then $\exists$ series with $\left|\mathcal{L}_{S_{i+1}}/\mathcal{L}_{S_i}\right| = 3$.

$|N| = w$ and $|Q| = z$ two affine Steiner loops

$\mathcal{Q}(N)$ the set of $w \times w$ latin squares with coefficients in the set N.

$\Phi : Q \times Q \longrightarrow \mathcal{Q}(N)$, which maps the pair $(\bar{x}, \bar{y})$ to a latin square

$\Phi_{\bar{x},\bar{y}} : N \times N \longrightarrow N$, and fulfills the following conditions:

i) $\Phi_{\bar{y},\bar{x}}(y', x') = \Phi_{\bar{x},\bar{y}}(x', y')$;

ii) $\Phi_{\bar{0},\bar{0}}$ is the table of addition of N;

iii) $\Phi_{\bar{x},\bar{0}}(x', 0') = x'$;

iv) $\Phi_{\bar{x},-\bar{x}}(x', \Phi_{\bar{y},\bar{z}}(y', z')) = 0$ if and only if $\Phi_{-\bar{z},\bar{z}}(\Phi_{\bar{x},\bar{y}}(x', y'), z') = 0$;

v) $\Phi_{\bar{x},-\bar{x}}(x', \Phi_{\bar{x},\bar{x}}(x', x')) = 0$;

is called a *Steiner operator*.

N and Q two affine Steiner loops

$\Phi : Q \times Q \longrightarrow \mathcal{Q}(N)$ a Steiner operator.

$$(\bar{x}, x') + (\bar{y}, y') = (\bar{x} + \bar{y}, x' \oplus y'),$$

where we denote, with abuse of notation,

$x' \oplus y' = \Phi_{\bar{x},\bar{y}}(x', y')$,

then L is an affine Steiner loop of order $v = wz$,

having N as a normal subloop and such that L/N is isomorphic to Q.

Conversely, any affine Steiner loop L,

having a normal subloop N and a quotient loop $Q = L/N$

is isomorphic to the above one.

$N:$

$+$	-1	0	1
-1	1	-1	0
0	-1	0	1
1	0	1	-1

$Q:$

$+$	$\bar{z}$	$\bar{y}$	$\bar{x}$	Ω	$-\bar{x}$	$-\bar{y}$	$-\bar{z}$
$\bar{z}$	$-\bar{z}$	$\bar{x}$	$\bar{y}$	$\bar{z}$	$-\bar{y}$	$-\bar{x}$	Ω
$\bar{y}$	$\bar{x}$	$-\bar{y}$	$\bar{z}$	$\bar{y}$	$-\bar{z}$	Ω	$-\bar{x}$
$\bar{x}$	$\bar{y}$	$\bar{z}$	$-\bar{x}$	$\bar{x}$	Ω	$-\bar{z}$	$-\bar{y}$
Ω	$\bar{z}$	$\bar{y}$	$\bar{x}$	Ω	$-\bar{x}$	$-\bar{y}$	$-\bar{z}$
$-\bar{x}$	$-\bar{y}$	$-\bar{z}$	Ω	$-\bar{x}$	$\bar{x}$	$\bar{z}$	$\bar{y}$
$-\bar{y}$	$-\bar{x}$	Ω	$-\bar{z}$	$-\bar{y}$	$\bar{z}$	$\bar{y}$	$\bar{x}$
$-\bar{z}$	Ω	$-\bar{x}$	$-\bar{y}$	$-\bar{z}$	$\bar{y}$	$\bar{x}$	$\bar{z}$

$L:$

$\Phi_{\bar{z},\bar{z}}$	$\Phi_{\bar{z},\bar{y}}$	$\Phi_{\bar{z},\bar{x}}$	$\Phi_{\bar{z},\Omega}$	$\Phi_{\bar{z},-\bar{x}}$	$\Phi_{\bar{z},-\bar{y}}$	$\Phi_{\bar{z},-\bar{z}}$
$\Phi_{\bar{y},\bar{z}}$	$\Phi_{\bar{y},\bar{y}}$	$\Phi_{\bar{y},\bar{x}}$	$\Phi_{\bar{y},\Omega}$	$\Phi_{\bar{y},-\bar{x}}$	$\Phi_{\bar{y},-\bar{y}}$	$\Phi_{\bar{y},-\bar{z}}$
$\Phi_{\bar{x},\bar{z}}$	$\Phi_{\bar{x},\bar{y}}$	$\Phi_{\bar{x},\bar{x}}$	$\Phi_{\bar{x},\Omega}$	$\Phi_{\bar{x},-\bar{x}}$	$\Phi_{\bar{x},-\bar{y}}$	$\Phi_{\bar{x},-\bar{z}}$
$\Phi_{\Omega,\bar{z}}$	$\Phi_{\Omega,\bar{y}}$	$\Phi_{\Omega,\bar{x}}$	$\Phi_{\Omega,\Omega}$	$\Phi_{\Omega,-\bar{x}}$	$\Phi_{\Omega,-\bar{y}}$	$\Phi_{\Omega,-\bar{z}}$
$\Phi_{-\bar{x},\bar{z}}$	$\Phi_{-\bar{x},\bar{y}}$	$\Phi_{-\bar{x},\bar{x}}$	$\Phi_{-\bar{x},\Omega}$	$\Phi_{-\bar{x},-\bar{x}}$	$\Phi_{-\bar{x},-\bar{y}}$	$\Phi_{-\bar{x},-\bar{z}}$
$\Phi_{-\bar{y},\bar{z}}$	$\Phi_{-\bar{y},\bar{y}}$	$\Phi_{-\bar{y},\bar{x}}$	$\Phi_{-\bar{y},\Omega}$	$\Phi_{-\bar{y},-\bar{x}}$	$\Phi_{-\bar{y},-\bar{y}}$	$\Phi_{-\bar{y},-\bar{z}}$
$\Phi_{-\bar{z},\bar{z}}$	$\Phi_{-\bar{z},\bar{y}}$	$\Phi_{-\bar{z},\bar{x}}$	$\Phi_{-\bar{z},\Omega}$	$\Phi_{-\bar{z},-\bar{x}}$	$\Phi_{-\bar{z},-\bar{y}}$	$\Phi_{-\bar{z},-\bar{z}}$

$$\Phi_{\Omega,\Omega}: \quad \begin{array}{c|ccc} + & (\Omega,-1) & (\Omega,0) & (\Omega,1) \\ \hline (\Omega,-1) & (\Omega,1) & (\Omega,-1) & (\Omega,0) \\ (\Omega,0) & (\Omega,-1) & (\Omega,0) & (\Omega,1) \\ (\Omega,1) & (\Omega,0) & (\Omega,1) & (\Omega,-1) \end{array}$$

Choose a latin square on $\{-1, 0, 1\}$ for $\Phi_{\bar{z},\bar{z}}$ and $\Phi_{-\bar{z},-\bar{z}}$,

and apply condition v) to $\Phi_{\bar{z},-\bar{z}}$:

$$\begin{array}{c|c} \Phi_{\bar{z},\bar{z}} & \Phi_{\bar{z},-\bar{z}} \\ \hline \Phi_{-\bar{z},\bar{z}} & \Phi_{-\bar{z},-\bar{z}} \end{array} :$$

$+$	$(\bar{z},-1)$	$(\bar{z},0)$	$(\bar{z},1)$	$(-\bar{z},-1)$	$(-\bar{z},0)$	$(-\bar{z},1)$
$(\bar{z},-1)$	$(-\bar{z},-1)$	$(-\bar{z},1)$	$(-\bar{z},0)$	$(\Omega,0)$	$(\Omega,1)$	$(\Omega,-1)$
$(\bar{z},0)$	$(-\bar{z},1)$	$(-\bar{z},0)$	$(-\bar{z},-1)$	$(\Omega,-1)$	$(\Omega,0)$	$(\Omega,1)$
$(\bar{z},1)$	$(-\bar{z},0)$	$(-\bar{z},-1)$	$(-\bar{z},1)$	$(\Omega,1)$	$(\Omega,-1)$	$(\Omega,0)$
$(-\bar{z},-1)$	$(\Omega,0)$	$(\Omega,-1)$	$(\Omega,1)$	$(\bar{z},1)$	$(\bar{z},-1)$	$(\bar{z},0)$
$(-\bar{z},0)$	$(\Omega,1)$	$(\Omega,0)$	$(\Omega,-1)$	$(\bar{z},-1)$	$(\bar{z},0)$	$(\bar{z},1)$
$(-\bar{z},1)$	$(\Omega,-1)$	$(\Omega,1)$	$(\Omega,0)$	$(\bar{z},0)$	$(\bar{z},1)$	$(\bar{z},-1)$

$$\Phi_{\bar{z},\bar{y}}: \begin{array}{c|ccc} + & (\bar{y},-1) & (\bar{y},0) & (\bar{y},1) \\ \hline (\bar{z},-1) & (\bar{x},1) & (\bar{x},-1) & (\bar{x},0) \\ (\bar{z},0) & (\bar{x},0) & (\bar{x},1) & (\bar{x},-1) \\ (\bar{z},1) & (\bar{x},-1) & (\bar{x},0) & (\bar{x},1) \end{array},$$

$$\Phi_{\bar{z},-\bar{x}}: \begin{array}{c|ccc} + & (-\bar{x},-1) & (-\bar{x},0) & (-\bar{x},1) \\ \hline (\bar{z},-1) & (-\bar{y},1) & (-\bar{y},-1) & (-\bar{y},0) \\ (\bar{z},0) & (-\bar{y},0) & (-\bar{y},1) & (-\bar{y},-1) \\ (\bar{z},1) & (-\bar{y},-1) & (-\bar{y},0) & (-\bar{y},1) \end{array},$$

$$\Phi_{\bar{y},-\bar{x}}: \begin{array}{c|ccc} + & (-\bar{x},-1) & (-\bar{x},0) & (-\bar{x},1) \\ \hline (\bar{y},-1) & (-\bar{z},1) & (-\bar{z},-1) & (-\bar{z},0) \\ (\bar{y},0) & (-\bar{z},0) & (-\bar{z},1) & (-\bar{z},-1) \\ (\bar{y},1) & (-\bar{z},-1) & (-\bar{z},0) & (-\bar{z},1) \end{array}.$$

Let S be a STS(n). Then

$$\lambda_x = (\Omega, x, -x)(v_1, v_2, v_3, ..., v_j)(-v_j, \ldots, -v_3, -v_2, -v_1)\ldots$$

$$\lambda_x = (\Omega, x, -x)\sigma_x.$$

If σ_x has the $(v_1, v_2, v_3, ..., v_j)$-cycle, then the STS has the following blocks:

$\{x, v_1, -v_2\}$, $\{x, v_2, -v_3\}$, . . . , $\{x, v_{j-1}, -v_j\}$, $\{x, v_j, -v_1\}$.

Thus σ_x has also the j-cycle

$(-v_j, \ldots, -v_3, -v_2, -v_1)$

If $\mathcal{S}$ is a simple STS(n) containing a Veblen point V, then

$$\mathcal{L}_{\mathcal{S}} = A_n / A_{n-1}$$

1) V is the neutral element Ω of $\mathcal{L}_{\mathcal{S}}$ if and only if, for any $x \neq \Omega$,

$$\rho_x = (\Omega, x, -x)(p_1, q_1)(-q_1, -p_1) \cdots (p_{\frac{n-3}{4}}, q_{\frac{n-3}{4}})(-q_{\frac{n-3}{4}}, -p_{\frac{n-3}{4}}).$$

2) If $V \neq \Omega$, then

$$\begin{cases} \rho_V = (\Omega, x, -x)(p_1, q_1)(-q_1, -p_1) \cdots (p_{\frac{n-3}{4}}, q_{\frac{n-3}{4}})(-q_{\frac{n-3}{4}}, -p_{\frac{n-3}{4}}) \\ \rho_{-V} = (\Omega, -x, x)(p_1, -q_1)(q_1, -p_1) \cdots (p_{\frac{n-3}{4}}, -q_{\frac{n-3}{4}})(q_{\frac{n-3}{4}}, -p_{\frac{n-3}{4}}). \end{cases}$$

Let $\mathcal{S}$ be a STS(13). Then

$$\mathcal{L}_{\mathcal{S}} = A_{13}/A_{12}$$

S_1 and S_2 the two non-isomorphic STS(13) as they are defined in [Limbos], p. 152-153. In S_1 take $\lambda_1 = (0,1,4)(2,7,3,10,6)(5,8,11,9,12)$, in S_2 take $\lambda_2 = (0,1,4)(2,12,5,10,6)(3,8,11,9,7)$.

In both cases, $\mathcal{L}_{\mathcal{S}} = A_{13}/A_{12}$ by Jordan's theorem.

Let $\mathcal{S}$ be a simple STS(n), such that $n \neq \frac{q^a-1}{q-1}$, for any prime power q.

Then $\mathcal{L}_{\mathcal{S}} = A_n/A_{n-1}$.

This follows from a celebrated theorem by Guralnik.