A REDUCTION THEOREM FOR NONSOLVABLE FINITE GROUPS

(joint w. F. Leinen & O. Puglisi).

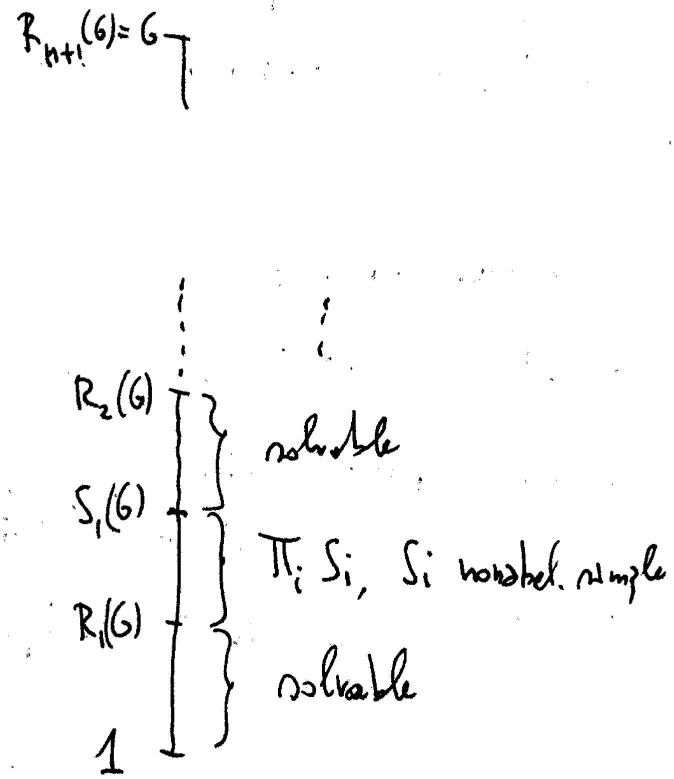
Let G be a finite group.

$R(G) := \langle H \mid H$ solvable normal subgp. of $G \rangle$

$S(G) := \langle K \mid K$ minimal normal nonabelian subgp. of $G \rangle$

The "R-S-series" of G:

$$\begin{cases} R_1(G) = R(G) \\ \dfrac{S_i(G)}{R_i(G)} = S\left(\dfrac{G}{R_i(G)}\right), \ \forall i \geq 1 \\ \dfrac{R_{i+1}(G)}{S_i(G)} = R\left(\dfrac{G}{S_i(G)}\right), \ \forall i \geq 1 \end{cases}$$



If $G = R_{n+1}(G) > R_n(G)$, then $n :=$ the NONSOLVABLE LENGTH of $G = \lambda(G)$

- $\lambda(G) = 0 \iff G$ is solvable
- $\lambda(G) = 1 \Leftarrow G$ (direct product of) simple / almost simple / quasisimple gps.
- $\lambda(\underbrace{(\ldots(A_5 \wr A_5) \wr A_5) \wr \ldots \wr A_5}_{n\text{-times}}) = n$.

- E. Khukhro & P. Shumyatsky (2015) $(\lambda(G), \lambda_p(G)$ also$)$.

Theorem (F. L. P. 2019)

Every finite group of nonsolvable length n contains an n-RAREFIED subgroup.

n-RAREFIED GROUPS:

G s.t. $\lambda(G) = n$ & the following holds:

1) $R_1(G) = \phi(G)$ & $\frac{R_{i+1}(G)}{S_i(G)} = \phi\left(\frac{G}{S_i(G)}\right) \quad \forall i \geq 1$;

2) Each $\frac{S_i(G)}{R_i(G)}$ is the unique minimal normal subgp. in $\frac{G}{R_i(G)}$, $\forall i \geq 1$

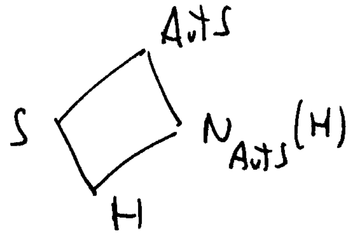
3) The components of each $\frac{S_i(G)}{R_i(G)}$ lie in

$$\mathcal{L} = \left\{ PSL_2(q), \text{ for } q \in \{2^r, 3^r, p^{2^a}\}, \text{ with } r, p \in \mathbb{P},\ 2 \neq p,\ a \geq 0;\ PSL_3(3),\ {}^2B_2(2^r),\ r \in \mathbb{P} \right\}.$$

Note that $\mathcal{L} \supset \mathcal{M} = \{\text{minimal simple gps}\}$, $(A_6 \in \mathcal{L} \setminus \mathcal{M})$.

Lemma. Assume that S is a finite simple (nonabelian) group.
If $S \notin \mathcal{L}$ then $\exists H \underset{+}{<} S$ s.t.

1) $\lambda(H) = 1$
2) $H = N_S(H)$
3) $Aut(S) = S \cdot N_{Aut(S)}(H)$



Applications of the Thm

- $\lambda(G) \leq \max\{ l_2(A) \mid A \text{ solvable subgp of } G\}$.
- If $\lambda(G) \geq 1$, then $\exists H \leq G$, H 2-generated s.t. $\lambda(H) = \lambda(G)$.
- $\max\{\lambda(G) \mid G \leq Sym(n)\} = \lfloor \log_5(n) \rfloor$

  $\max\{\lambda(G) \mid G \leq GL_n(\mathbb{F})\} = 1 + \lfloor \log_5(n/2) \rfloor$ (if $|\mathbb{F}| \geq 4$
  when $|\mathbb{F}| \leq 3$ bounds)

Conjecture I. Let G s.t. $\lambda(G) = n$ and w a reduced word in $k \geq 2$ variables.
If length$(w) = n$, then $w(G) \neq 1$.

Conjecture II ("The Puglisi Alternative"). Any fin. gen. profinite group either contains a free subgroup or admits a prosolvable group which is "big enough".