On the connection between Mealy and Moore automata via d-adic dynamics

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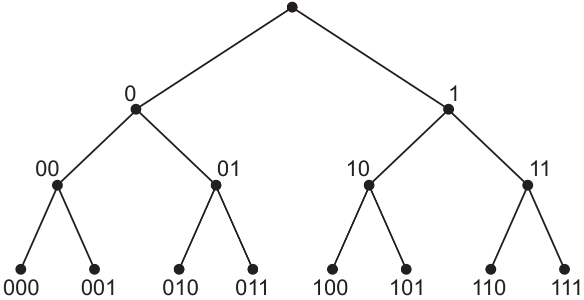
The Rooted d-ary Tree

- Let $X = \{0, 1, \ldots, d-1\}$ be a finite alphabet and X^* – the set of all finite words over X.
- Label the vertices of a rooted d-ary tree by elements from X^*



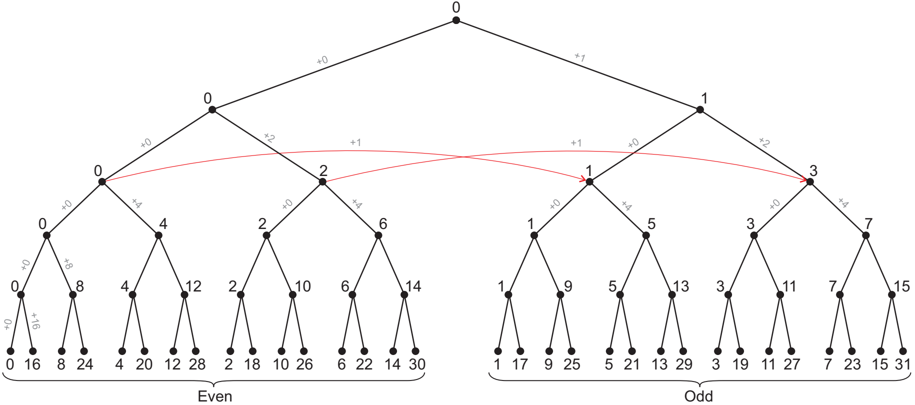
- We will consider automorphisms and endomorphisms of this tree

The n^{th} Level Viewed as $\mathbb{Z}/(d^n\mathbb{Z})$

- Identify the n^{th} level of X^* with the ring $\mathbb{Z}/(d^n\mathbb{Z})$ by

$$X^n \ni x_0x_1\ldots x_{n-1} \longleftrightarrow x_0 + dx_1 + \cdots + d^{n-1}x_{n-1} \in \mathbb{Z}/(d^n\mathbb{Z})$$

- The boundary X^∞ of the tree is then identified with the ring $\mathbb{Z}_d$ of d-adic integers.



Proposition

a) A transformation $f : \mathbb{Z}_d \to \mathbb{Z}_d$ *defines an endomorphism of* X^* *if and only if it is 1-Lipschitz*
b) A transformation $f : \mathbb{Z}_d \to \mathbb{Z}_d$ *defines an automorphism of* X^* *if and only if it is an isometry. Hence* $Aut(X^*) = Isom(\mathbb{Z}_d)$.

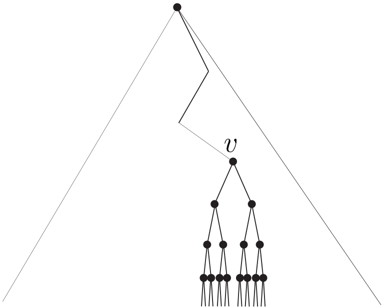
How to describe endomorphisms?

Definition

Let g be an endomorphism of X^* and $v \in X^*$. For any word $w \in X^*$, we have $g(vw) = g(v)w'$ for some $w' \in X^*$. The map $g|_v : X^* \to X^*$ given by

$$g|_v(w) = w'$$

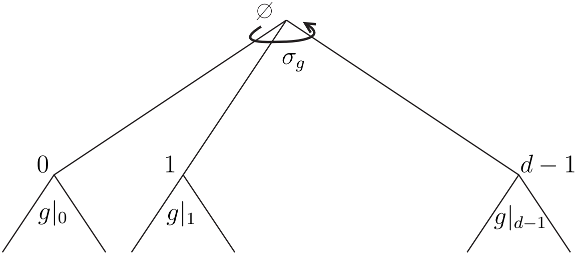
defines an endomorphism of X^* and is called the section of g at v.



The Action of an Endomorphism

Thus, each endomorphism of X^* can be described as

$$g = (g|_0, g|_1, \ldots, g|_{d-1})\sigma_g$$



Definition

An endomorphism g of X^* is called finite state if $|\{g|_v : v \in X^*\}| < \infty$.

Maps $\mathbb{Z}_d \to \mathbb{Z}_d$?

Any continuous $f\colon \mathbb{Z}_d \to \mathbb{Z}_d$ can be described by its van der Put series:

$$f(x) = \sum_{n \geq 0} B_n^f \chi(n, x),$$

$B_n^f \in \mathbb{Z}_d$ are computed as:

$$B_n^f = \begin{cases} f(n), & \text{if } \ 0 \leq n < d, \\ f(n) - f(n_{-}), & \text{if } \ n \geq d, \end{cases}$$

where for $n = x_0 + x_1 d + \cdots + x_t d^t$ with $x_t \neq 0$ we define $n_{-} = x_0 + x_1 \cdot d + \cdots + x_{t-1} d^{t-1}$, and

$$\chi(n, x) = \begin{cases} 1, & d\text{-ary expansion of } n \text{ is a prefix of } x, \\ 0, & \text{otherwise.} \end{cases}$$

Denote $\lfloor \log_d n \rfloor = t$.

Proposition (Anashin-Khrennikov-Yurova'11)

A map $f : \mathbb{Z}_d \to \mathbb{Z}_d$ is 1-Lipschitz (i.e. defines an endomorphism of X^*) $\Leftrightarrow$

$$f(x) = \sum_{n \geq 0} b_n^f d^{\lfloor \log_d n \rfloor} \chi(n, x).$$

b_n^f – reduced van der Put coefficients.

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Theorem (Anashin'12)

An endomorphism $f : \mathbb{Z}_d \to \mathbb{Z}_d$ is finite state $\Leftrightarrow$

(a) $\{b_n^f\}_{n \geq 0}$ *is d-automatic*

(b) b_n^f *is preperiodic for each n (equivalently $b_n^f \in \mathbb{Z}_d \cap \mathbb{Q}$).*

Automatic Sequences

Definition

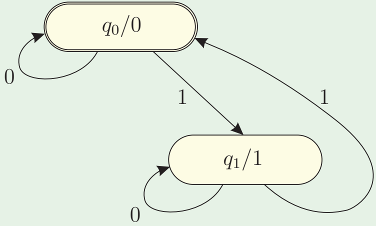
We say the sequence $(a_n)_{n\geq 0}$ over a finite alphabet A is d-automatic if it can be generated by a finite Moore automaton. To compute a_n:

- write the (reversed) d-ary expansion w of n
- feed w to the automaton
- $a_n =$ label of the ending state

Example

The Thue-Morse sequence 0110 1001 10010110 1001011001101001 . . .

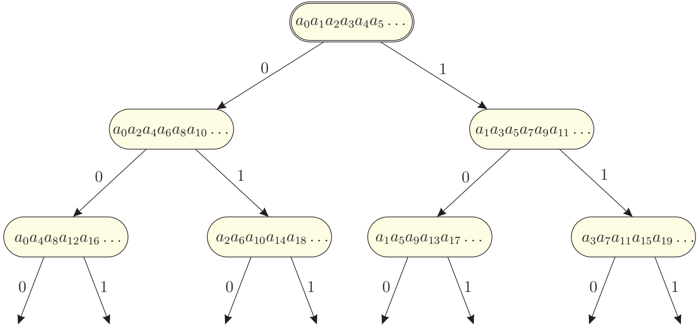
n	$[n]_2$	t_n	n	$[n]_2$	t_n
0	0	0	4	001	1
1	1	1	5	101	0
2	01	1	6	011	0
3	11	0	7	111	1



Alternative definition via subsequences

Theorem

A sequence $(a_n)_{n\geq 0}$ over an alphabet A is d-automatic if and only if the collection of its subsequences of the form $\{(a_{j+n\cdot d^i})_{n\geq 0} \mid i \geq 0, 0 \leq j < d^i\}$, called the d-kernel, is finite.



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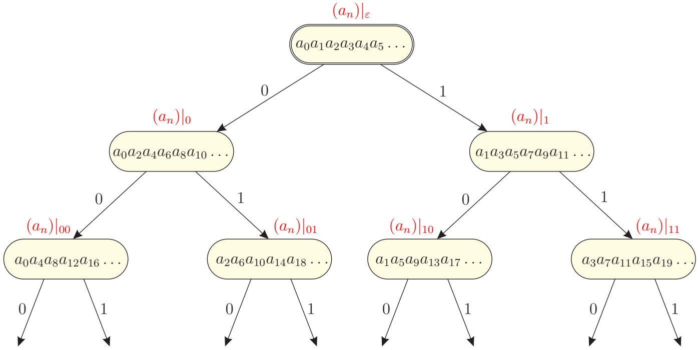

Proposition

A sequence $(a_n)_{n\geq 0}$ *over an alphabet* A *is* d*-automatic if and only if the collection of its sections* $\left\{(a_n)_{n\geq 0}\big|_v : v \in X^*\right\}$ *is finite.*

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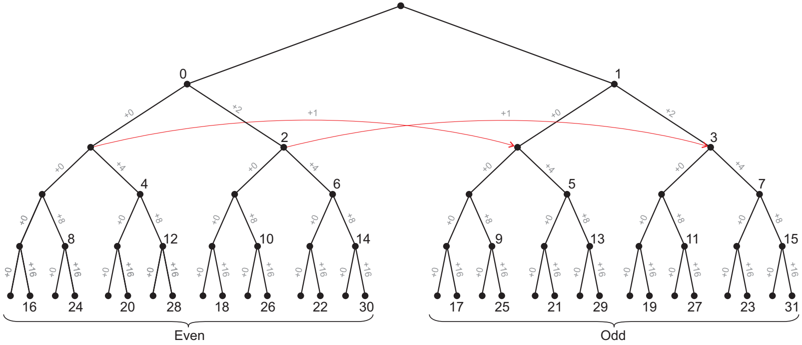
One can visualize it geometrically using the portrait of a sequence on a rooted tree.

First, label SOME vertices of the tree by integers:

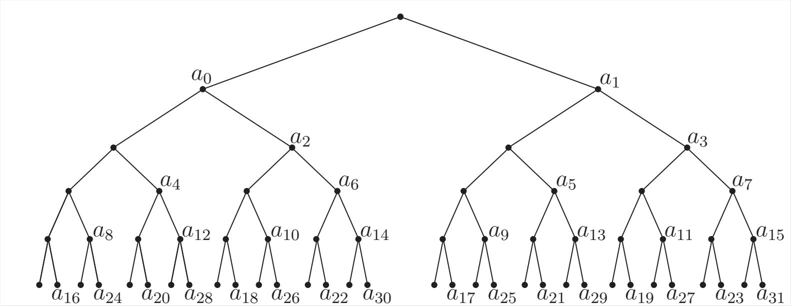
$$n \mapsto \text{reversed } d\text{-ary expansion of } n$$

$$\text{E.g. } 6 \mapsto 011$$



Definition

The portrait of a sequence $(a_n)_{n\geq 0}$ over an alphabet A is a d-ary rooted tree X^* where the vertex n is labelled by a_n.



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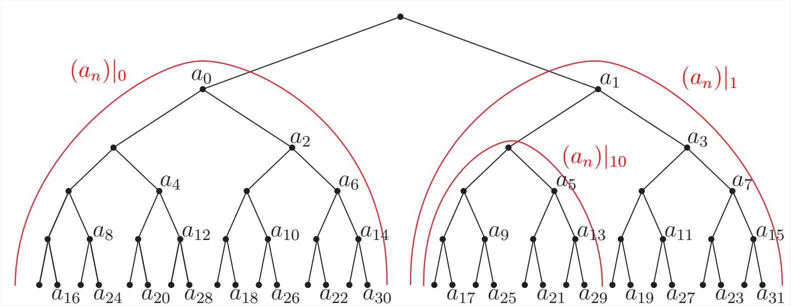
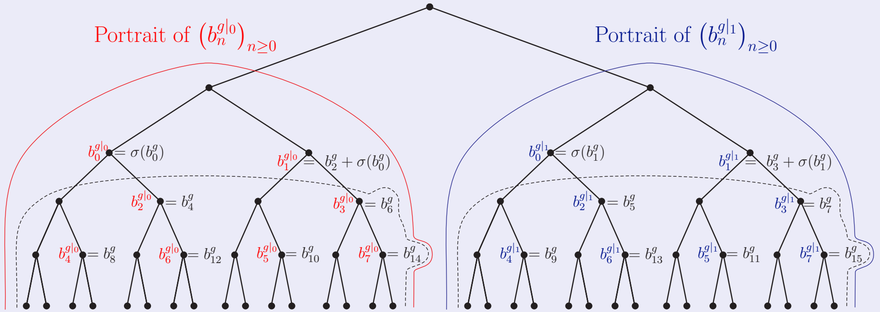


Theorem (Grigorchuk-S.'19+)

Suppose $g \in \mathrm{End}\, X^*$. Then the reduced van der Put coefficients of the sections $g|_x$ are defined as follows:



where for $b \in \mathbb{Z}_d$ we denote by $\sigma(b) = \frac{b - (b \bmod d)}{d}$ the shift map on $\mathbb{Z}_d$.

This gives a CONSTRUCTVE proof of Anashin's automaticity result.

Theorem (Grigorchuk,S.)

- *Let $g \in End(X^*)$ be defined by a finite Mealy automaton $\mathcal{A}$. Then the underlying oriented graph of the Moore automaton $\mathcal{B}$ defining $(b_n^g)_{n\geq 0}$ (possibly non-minimized) covers the underlying oriented graph of $\mathcal{A}$.*
- *Let $g \in End(X^*)$ be such that $(b_n^g)_{n\geq 0}$ is defined by finite Moore automaton $\mathcal{B}$. Then the underlying oriented graph of the Mealy automaton $\mathcal{A}$ defining g (possibly non-minimized) covers the underlying oriented graph of $\mathcal{B}$.*

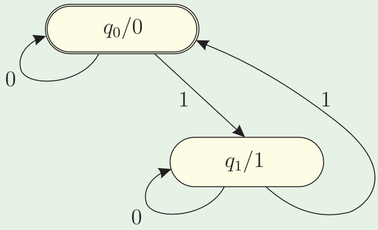
Automata associated to the Thue-Morse sequence

Example

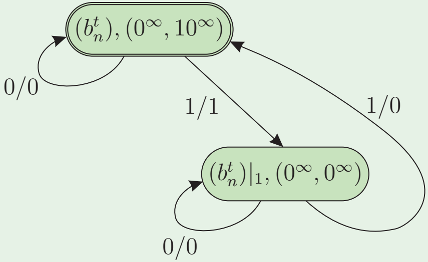
The Thue-Morse sequence

$$0110\ 1001\ 10010110\ 1001011001101001\ldots$$

Moore



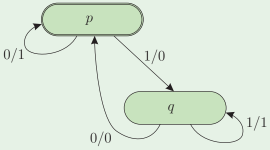
Mealy



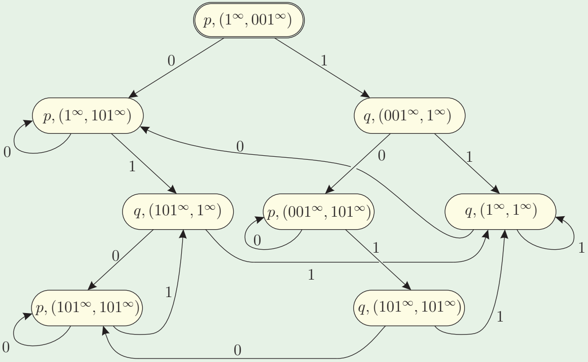
Automata associated to the Lamplighter group

Example

Mealy



Moore



Thank You!