Circulant automata synchronize with high probability

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Overview

Definitions

Complete Deterministic Finite Automaton (DFA)

$$\mathcal{A}_{n,k} := \langle \underbrace{Q}_{\text{Finite set of } states}, \overbrace{\Sigma}^{\subset \mathcal{P}\left(Q^Q\right);\ letters} \rangle$$

$$|Q| = n;\ |\Sigma| = k$$

$\mathcal{A}$ synchronizes if $\exists\, w$ word s.t. $|Qw| = 1$

Černý conjecture

If $\mathcal{A}_n$ synch. $\Rightarrow \exists$ synch. w of length $\leq (n-1)^2$

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Recent related results

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- (1) Random automata synch. *with high probability* (whp)?
- (2) Synch. automata have synch. words of length $\leq (n-1)^2$ whp?

- [SZ10] Results on large alphabets;
- [Ber13] $\mathbb{P}[\mathcal{A}_{n,k} \text{ synchronizes}] = 1 - \mathcal{O}(n^{-k/2})$;
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Circular automata

$$\mathcal{A}_m(\mathbf{b}) := \langle \mathbb{Z}_m, \{ \underbrace{a}_{i \mapsto i+1}, \overbrace{\mathbf{b}}^{\mathsf{Uniform}\left({\mathbb{Z}_m}^{\mathbb{Z}_m}\right)} \} \rangle$$

Theorem ([Pin78])

$\mathbb{P}\left[\mathcal{A}_p(\mathbf{b}) \text{ synchronizes}\right] = 1 - \frac{p!}{p^p} = 1 - \Theta\left(\frac{\sqrt{p}}{e^p}\right)$, p *prime*

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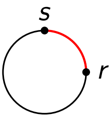
Proof sketch (1)

$\left| r - s \right|_n :=$ 

$$\begin{bmatrix} \left|b_0 - b_1\right|_n & \left|b_1 - b_2\right|_n & \cdots & \left|b_k - b_{k+1}\right|_n & \cdots & \left|b_{n-1} - b_0\right|_n \\ \left|b_0 - b_2\right|_n & \left|b_1 - b_3\right|_n & \cdots & \left|b_k - b_{(k+2)_n}\right|_n & \cdots & \left|b_{n-1} - b_1\right|_n \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \left|b_0 - b_i\right|_n & \left|b_1 - b_{1+i}\right|_n & \cdots & \left|b_k - b_{(k+i)_n}\right|_n & \cdots & \left|b_{n-1} - b_{(n-1+i)_n}\right|_n \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \left|b_0 - b_{\lfloor \frac{n}{2} \rfloor}\right|_n & \left|b_1 - b_{1+\lfloor \frac{n}{2} \rfloor}\right|_n & \cdots & \left|b_k - b_{(k+\lfloor \frac{n}{2} \rfloor)_n}\right|_n & \cdots & \left|b_{n-1} - b_{(n-1+\lfloor \frac{n}{2} \rfloor)_n}\right|_n \end{bmatrix}$$

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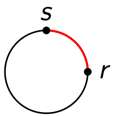
$\left|r - s\right|_n :=$ 

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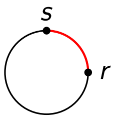

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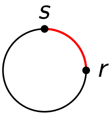

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$$\begin{aligned}\mathbb{P}\left[\mathcal{A}_n(\mathbf{b}) \text{ synch.}\right] &\geq \mathbb{P}\left[\mathcal{E}_{\text{row}} \cap \mathcal{E}_{\text{zero}}\right] \\ &= 1 - \mathbb{P}\left[(\mathcal{E}_{\text{row}})^c \cup (\mathcal{E}_{\text{zero}})^c\right] \\ &\geq 1 - \mathbb{P}\left[(\mathcal{E}_{\text{row}})^c\right] - \mathbb{P}\left[(\mathcal{E}_{\text{zero}})^c\right]\end{aligned}$$

- $(\mathcal{E}_{\text{row}})^c =$ **not** every row has "many" elements;
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- Row i has a "large" set of i.i.d entries;
- $\mathbb{P}\left[\left(\mathcal{E}_{\text{row i}}\right)^c\right] \overset{\text{McDiarmid}}{\leq} e^{-\Theta(n)}$;
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$(\mathcal{E}_{\text{zero}})^c =$ **"few"** rows with at least one zero;

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- $\times$ McDiarmid, $\times$ Azuma;
- $\times$ $\mathbb{V}[\mathbb{D}]$;
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Future work

$$\mathcal{A}_n(\mathbf{b}) := \langle \mathbb{Z}_n, \{a, \overbrace{\mathbf{b}}^{\text{Uniform}\left({\mathbb{Z}_n}^{\mathbb{Z}_n}\right)}\}\rangle$$

- "a" finite number of pairwise disjoint cycles of almost-equal length;
- $\mathbb{P}\left[\mathcal{A}_n(\mathbf{b}) \text{ synch.}\right] \stackrel{?}{=} 1 - \mathcal{O}(e^{-\epsilon n})$, for some $\epsilon > 0$ where $a(i) = i + 1$?

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$$\mathcal{A}_n(\mathbf{b}) := \langle \mathbb{Z}_n, \{a, \overbrace{\mathbf{b}}^{\text{Uniform}\left({\mathbb{Z}_n}^{\mathbb{Z}_n}\right)}\}\rangle$$

- "a" finite number of pairwise disjoint cycles of almost-equal length;
- $\mathbb{P}\left[\mathcal{A}_n(\mathbf{b}) \text{ synch.}\right] \stackrel{?}{=} 1 - \mathcal{O}(e^{-\epsilon n})$, for some $\epsilon > 0$ where $a(i) = i + 1$?

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