On a Conjecture in Artin groups

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Introduction

Let $m, n \in \mathbb{N}$ ($\mathbb{N}$ - the natural numbers), $m \geq 0$, $n \geq 1$ and let Γ be a simple graph without loops, with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $E = \{e_1, \ldots, e_m\}$. Label the edges by natural numbers via a labelling function $\lambda : E \to \mathbb{N}$. Denote $\lambda(e) = n_{ij}$, where e connects v_i to v_j, $i \neq j$ ($n_{ij} = n_{ji}$). For every such labelled graph correspond a group presentation $\mathcal{P}(\Gamma) = \langle x_1, \ldots, x_n \mid R_e \ ,\ e \in E \rangle$ such that $R_e = (x_i x_j)^{\frac{1}{2} n_{ij}} \left(x_i^{-1} x_j^{-1} \right)^{\frac{1}{2} n_{ij}}$ for n_{ij} even and $R_e = (x_i x_j)^{\frac{n_{ij}-1}{2}} x_i \cdot (x_j^{-1} x_i^{-1})^{\frac{n_{ij}-1}{2}} x_j^{-1}$ for n_{ij} odd. The group defined by $\mathcal{P}(\Gamma)$ is denoted by $A(\Gamma)$. It is the *Artin group defined by* Γ. We say that $A(\Gamma)$ is *irreducible* if $A(\Gamma)$ is not the direct sum of two Artin subgroups $A_1 = \langle X_1 \rangle$ and $A_2 = \langle X_2 \rangle$ with $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$.

Examples

1. $n_{ij} = 0$ for every i, j, $1 \leq i < j \leq n$.
   Then A is the free group on X.

2. $n_{ij} = 2$ for every i, j, $1 \leq i < j \leq n$.
   Then A is the free abelian group on X.

3. $n_{ij} \in \{0, 2\}$ for every i, j, $1 \leq i < j \leq n$.
   Then A is called right angled.

To each Artin group A there corresponds a Coxeter group W_A, obtained by adding the relators x_i^2, $i = 1, \ldots, n$. An Artin group is called *spherical* if W_A is a finite group. The graphs of the spherical Artin groups are classically known.

Example $X = \{x_1, x_2, x_3\}$

Then $A(\Gamma)$ is of finite type (or spherical) if and only if Γ is a triangle with vertices x_1, x_2 and x_3 such that

$$\frac{1}{n(x_1, x_2)} + \frac{1}{n(x_2, x_3)} + \frac{1}{n(x_3, x_1)} > 1$$

The Conjecture

Let A be an irreducible Artin group. If A is not of finite type then the center of A is $\{1\}$.

This conjecture is known to be true for right angled Artin groups and Artin groups with cohomological dimension ≤ 2.

In the present talk we confirm this conjecture for Artin groups under an assumption which is stronger than not being spherical.

Definition Let A be an Artin group, $A = A(\Gamma)$. Call A *pointwise-spherical* if every vertex v of Γ is contained in a spherical triangle, i.e. there are distinct vertices u, w in Γ such that

$$\frac{1}{n_{u,v}} + \frac{1}{n_{v,w}} + \frac{1}{n_{w,u}} > 1$$

Say that A is *not pointwise-spherical* if Γ contains a vertex which is not contained in any spherical triangle.

If A is 2-dimensional then clearly A is not pointwise-spherical. However, if A is not pointwise-spherical it may have any cohomological dimension.

Theorem A

Our main theorem is the following:

Theorem A *Let A be a not pointwise-spherical irreducible Artin group.*
Then $Z(A) = 1$.

Notation and preliminary notions

Notation

We denote by $|W|$ the length of a word $W \in F(x_1, \ldots, x_n)$ and by $||W||$ its syllable length (also called free-product length). Thus $|x_1^2 x_3^{-3} x_5| = 2 + 3 + 1 = 6$ and $||x_1^2 x_3^{-3} x_5|| = 3$.
We denote by $Supp(W)$ the letters of X which occur in W or W^{-1}.

Relative Presentations

Every group G has a free presentation $\mathcal{P} = \langle X \mid \mathcal{R} \rangle$ by a set of generators X and a set of cyclically reduced defining relators $\mathcal{R} \subseteq F(X)$, where $F(X)$ is the free group, freely generated by X and $G \cong F(X)/N$, where N is the normal closure of $\mathcal{R}$ in $F(X)$ and $\mathcal{R}$ cyclically closed.

In certain cases it is convenient to ignore a part of the defining relators and to emphasize the others. This can be done when we can divide X and $\mathcal{R}$ into subsets $X = X_1 \dot{\cup} X_2$ and $\mathcal{R} = \mathcal{R}_1 \dot{\cup} \mathcal{R}_2$ such that $\mathcal{R}_1 \subseteq F(X_1)$ and $\mathcal{R}_2 \not\subseteq F(X_1)$. Then we ignore $\mathcal{R}_1$ and emphasize $\mathcal{R}_2$ by considering the quotient of $H * F(X_2)$ by the normal closure of $\mathcal{R}_2$ in $H * F(X_2)$, where $H = F(X_1)/N_1$ and N_1 is the normal closure of $\mathcal{R}_1$ in $F(X_1)$. This gives a *presentation of G relative to H*. More precisely,

Definition A *relative presentation* $\mathbb{P}$ consists of a group H, a set Y, $Y \cap H = \emptyset$ and a set $\mathcal{R}'$ of cyclically reduced words in $H * F(Y)$ which are not in H. We denote

$$\mathbb{P} = \langle H, Y | \mathcal{R}' \rangle \qquad (1')$$

The group defined by $\mathbb{P}$ is $H * F(Y)/\langle\langle \mathcal{R}' \rangle\rangle$, where $\langle\langle \mathcal{R}' \rangle\rangle$ is the normal closure of $\mathcal{R}'$ in $H * F(Y)$.

Definition

$\mathbb{P}$ is called *aspherical* if there are no nontrivial identities among the relations.

Example 3

$X = \{a, b, c\}$, $\mathcal{R} = \{R_1, R_2, R_3\}$, where $R_1 = aba^{-1}b^{-1}$, $R_2 = aca^{-1}c^{-1}$, $R_3 = bcb^{-1}c^{-1}$. Here $G \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

We have the following identity relations
$R_1 R_2^b R_3 (R_1^{-1})^c R_2^{-1} (R_3^{-1})^a$, where $R^x = xRx^{-1}$

3. Proof of the Main Result

3.1 **Klyachko's Words and asphericity of $\mathbb{L}$**

In this Section we are going to apply the following result of A.A. Klyachko, concerning the center of relative presentations by which we show that in most of the case $Z(A) = 1$. (Proposition 3.2). We also consider the remaining cases, where Klyachko's result cannot be applied (Proposition 3.3).

Klyachko's Lemma

Let $\mathbb{P} = \langle H, X_1 | \mathcal{R} \rangle$ *be a relative presentation,* $H \neq 1, X_1 \neq \emptyset$.
Suppose that there exists a word $W \in H * F(X_1)$ *such that*

$$W \text{ is not a proper power in } H * F(X_1) \qquad (*)$$

and there exists a positive integer l *such that*

$$W^l \text{ is not conjugate in } H * F(X_1) \text{ to any element of } H \cup \mathcal{R} \cup \mathcal{R}^{-1} \qquad (**)$$

where $\mathcal{R}^{-1} = \{R^{-1} | R \in \mathcal{R}\}$

Let $\mathbb{L}$ *be the relative presentation obtained from* $\mathbb{P}$ *by adding* W *as a relator* (2)

If $\mathbb{L}$ *is aspherical, then*

$$Z(G) = 1, \text{ where } G \text{ is the group presented by } \mathbb{P} \qquad (***)$$

3.3 Proof of the Main Result

We start the proof with a subdivision of X_0 according to the relations between its elements and elements of X_1. Thus for an element $s \in X_1$ define

$$C_s = \{x \in X_0 | n(x,s) = 2\},$$
$$K_s = \{x \in X_0 | n(x,s) = 0\} \text{ and}$$

$$T_s = \{x \in X_0 | w(x,s) \geq 3\} \text{ Then}$$

$$X_0 = C_s \cup T_s \cup K_s \tag{2}$$

Propositon 3.2 *If $X_0 \neq K_s \cup C_s$ then there exists a K-word for $\mathbb{P}$.*

The proof is by constructing a long word which satisfies conditions $(*)$ and $(**)$ and application of combinatorial curvature calculations.

$K(v) = 2 - d(v) + 2\left(\sum \frac{1}{d(D_i)}\right)$

Proposition 3.3

Suppose $X_0 = C_s \cup K_s$. Then one of the following holds,

1. *A has a K-word (hence $Z(A) = 1$)*
2. *$C_{s_i} = \langle c \rangle$ for every $s_i \in X_1$ and $Z(A) = Z_\Gamma(A) = \langle c \rangle$*
3. *$Z(A) = 1$.*

Proof of Proposition 3.3

Suppose 1 does not hold. Then $\mathbb{L}$ is not aspherical.

<u>Case 1</u> $X_1 = \{s\}$

We have $A = \langle s, K_s, C_s \rangle$. Taking into account the fact that C_s is free, it follows by standard group theory and general results on Artin groups that

$$A = P_1 *_Q P_2 * \cdots * P_k \quad , \quad Q = \langle s, K_s \rangle \rangle \quad , \quad P_i = \langle Q, c_i \rangle \quad (3)$$

$$C_s = \{c_1, \ldots, c_k\}$$

Hence, if $k \geq 2$, then
$Z(A) \subseteq Z(Q) = Z(\langle s, K_s \rangle) = Z(\langle s \rangle * K_s) = 1.$

If $k = 1$ then $A = \langle s, c_1, K_s \rangle$. It follows from the assumption that $\mathbb{L}$ is not aspherical that for every $a \in K_s$.

$$a^i c_1^\alpha a^j c_1^\beta = 1,$$

By general results in Artin groups this implies that $[a, c_1] = 1$, for any $a \in K_s$.

Hence c_1 commutes with K_s and certainly with s. Therefore $A = \langle s, K_s \rangle \oplus \langle c_1 \rangle = (\langle s \rangle * K_s) \oplus \langle c_1 \rangle$, hence $Z(A) = Z\left(\langle s \rangle * K_s\right) \oplus Z(\langle c_1 \rangle) = \langle c_1 \rangle$.

Case 2 $|X_1| \geq 2$

Then similar arguments (in a different order) show that either $Z(A) = 1$ or $Z(A) = Z_\Gamma(A) = \langle c \rangle$, for some $c \in X_0$.

Proof of the Main Result

By Klyachko's Lemma (Lemma 3.1) if $\mathbb{P}$ has a K-word then $Z(A) = 1$. By Propositions 3.2 and 3.3 either $\mathbb{P}$ has a K-word or $Z(A) = Z_\Gamma(A) = \langle c \rangle$, for some $c \in X_0$. Hence if $Z(A) \neq 1$ then $Z(A) = Z_\Gamma(A) = \langle c \rangle$.

□