On the power pseudovarieties of some pseudovarieties of completely simple semigroups

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The celebrated equality of pseudovarieties **PG** = **BG**

In the early 90th of the last century, the nowadays celebrated equality of pseudovarieties **PG** = **BG** has been established, where **PG** is the pseudovariety of monoids generated by the power monoids of all finite groups, and **BG** is the pseudovariety of all block groups.

The celebrated equality of pseudovarieties $\mathbf{PG} = \mathbf{BG}$

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Actually a longer sequence of equalities of pseudovarieties, namely the sequence of equalities

$$\mathbf{PG} = \mathbf{J} * \mathbf{G} = \mathbf{J} \circledcirc\!\!\!\!\!\;m\; \mathbf{G} = \mathbf{BG}$$

has been obtained alongside. Here $\mathbf{J} * \mathbf{G}$ is the pseudovariety of monoids generated by the family of all semidirect products of $\mathcal{J}$-trivial monoids by groups, and $\mathbf{J} \circledcirc\!\!\!\!\!\;m\; \mathbf{G}$ is the pseudovariety generated by the Malćev product of the pseudovarieties $\mathbf{J}$ and $\mathbf{G}$.

The resources for the equality **PG** = **BG**

The mentioned equality of pseudovarieties has been obtained as a consequence of the deep results achieved in the following two concurrent papers:

- Ash, C. J.: Inevitable graphs: A proof of the type II conjecture and some related decision procedures. Internat. J. Algebra Comput. **1**, 127–146 (1991)
- Henckell, K., Rhodes, J.: The theorem of Knast, the **PG** = **BG** and Type-II conjectures. In: Monoids and Semigroups with Applications, Proceedings of the Workshop on Monoids held at the University of California, Berkeley, Calif., 1989, pp. 453–463. World Sci. Publ., Singapore (1991)

The equality of pseudovarieties $\mathbf{PH} = \mathbf{J} * \mathbf{H}$

The aforementioned equalities of pseudovarieties $\mathbf{PG} = \mathbf{J} * \mathbf{G} = \mathbf{J} \textcircled{m} \mathbf{G}$ have been subsequently generalized several times.

Thus, at first, it has been shown that the equality of pseudovarieties $\mathbf{PH} = \mathbf{J} * \mathbf{H}$ holds for every pseudovariety $\mathbf{H}$ of groups which is closed under co-extensions by elementary abelian p-groups for some prime number p:

Steinberg, B.: Polynomial closure and topology. Internat. J. Algebra Comput. **10**, 603–624 (2000)

Steinberg, B.: A note on the equation $\mathbf{PH} = \mathbf{J} * \mathbf{H}$. Semigroup Forum **63**, 469–474 (2001)

The equality of pseudovarieties $\mathbf{J} * \mathbf{H} = \mathbf{J} ⓜ \mathbf{H}$

Further on, it has been shown that the equality of pseudovarieties $\mathbf{J} * \mathbf{H} = \mathbf{J} ⓜ \mathbf{H}$ holds for every non-trivial extension closed pseudovariety $\mathbf{H}$ of groups:

Steinberg, B.: Finite state automata: a geometric approach. Trans. Amer. Math. Soc. **353**, 3409–3464 (2001)

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Steinberg, B.: Finite state automata: a geometric approach. Trans. Amer. Math. Soc. **353**, 3409–3464 (2001)

Subsequently, it has been shown that the equality of pseudovarieties $\mathbf{J} * \mathbf{H} = \mathbf{J} \circledm \mathbf{H}$ holds for every non-trivial arborescent pseudovariety $\mathbf{H}$ of groups:

Steinberg, B.: Inverse automata and profinite topologies on a free group. J. Pure Appl. Algebra **167**, 341–359 (2002)

Locally extensible pseudovarieties **H** of groups

In order to proceed further, we need another important notion.

Proposition

For every pseudovariety **H** *of groups, the following two conditions are equivalent:*

- *For each group G in* **H***, there exists a prime number p such that* **H** *contains each group which arises as a co-extension of G by an elementary abelian p-group.*
- *For each group G in* **H***, there exists a non-trivial cyclic group C such that the wreath product $C \circ G$ also belongs to* **H***.*

Definition

A pseudovariety **H** of groups is called locally extensible if it satisfies any one of the equivalent conditions given in the above proposition.

The equalities of pseudovarieties $\mathbf{PH} = \mathbf{J} * \mathbf{H} = \mathbf{J} ⓜ \mathbf{H}$

Then the previous findings have been crowned by showing that the equalities of pseudovarieties $\mathbf{PH} = \mathbf{J} * \mathbf{H} = \mathbf{J} ⓜ \mathbf{H}$ do hold for every locally extensible pseudovariety $\mathbf{H}$ of groups:

- Auinger, K., Steinberg, B.: The geometry of profinite graphs with applications to free groups and finite monoids. Trans. Amer. Math. Soc. **356**, 805–851 (2004)
- Auinger, K., Steinberg, B.: Constructing divisions into power groups. Theoret. Comput. Sci. **341**, 1–21 (2005)
- Auinger, K., Steinberg, B.: On power groups and embedding theorems for relatively free profinite monoids. Math. Proc. Cambridge Philos. Soc. **138**, 211–232 (2005)

The power pseudovariety **PCS**

We are next concerned with the study of the power pseudovariety **PCS**, where **PCS** is the pseudovariety of semigroups generated by the power semigroups of all finite completely simple semigroups.

The first achievement in this direction consisted in the verification of the equality of pseudovarieties $\mathbf{PCS} = \mathbf{BG} * \mathbf{RZ}$, providing essentially a solution of the membership problem for the power pseudovariety **PCS**:

Steinberg, B.: A note on the power semigroup of a completely simple semigroup. Semigroup Forum **76**, 584–586 (2008)

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The next development needs the notion of aggregates of block groups.

Definition

A finite semigroup S is said to be an aggregate of block groups if all subsemigroups of S of the form aSb with $a, b \in S$ are block groups.

The notable equality of pseudovarieties **PCS** = **AgBG**

The subsequent remarkable success in the investigation of the power pseudovariety **PCS** consisted in the establishment of the notable equality of pseudovarieties **PCS** = **AgBG**, where **AgBG** is the pseudovariety of all aggregates of block groups:

Auinger, K.: On the power pseudovariety **PCS**. Publ. Mat. **56**, 449–465 (2012)

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Auinger, K.: On the power pseudovariety **PCS**. Publ. Mat. **56**, 449–465 (2012)

In fact, from the results in the above-mentioned paper more equalities of pseudovarieties ensue. The entire sequence of equalities

$$\mathbf{PCS} = \mathbf{J} * \mathbf{CS} = \mathbf{J} \,ⓜ\, \mathbf{CS} = \mathbf{AgBG}$$

actually holds. Here, as before, $\mathbf{J} * \mathbf{CS}$ is the pseudovariety generated by the family of all semidirect products of $\mathcal{J}$-trivial semigroups by completely simple semigroups, and $\mathbf{J} \,ⓜ\, \mathbf{CS}$ is the pseudovariety generated by the Mal'cev product of the pseudovarieties **J** and **CS**.

Generalizing the equalities $\mathbf{PCS} = \mathbf{J} * \mathbf{CS} = \mathbf{J} \circledm \mathbf{CS}$.

We wish to generalize the aforementioned equalities of pseudovarieties $\mathbf{PCS} = \mathbf{J} * \mathbf{CS} = \mathbf{J} \circledm \mathbf{CS}$. That is to say, we seek for a broader class of pseudovarieties $\mathbf{Q}$ of completely simple semigroups for which the equalities $\mathbf{PQ} = \mathbf{J} * \mathbf{Q} = \mathbf{J} \circledm \mathbf{Q}$ would hold. In doing so, we restrict our attention to the pseudovarieties $\mathbf{Q}$ of the form $\mathbf{CS}(\mathbf{H})$ consisting of all completely simple semigroups all of whose subgroups belong to a given pseudovariety $\mathbf{H}$ of groups. Guided by the results listed previously in this talk, it seems reasonable to expect that the mentioned equalities would hold for the pseudovarieties $\mathbf{Q}$ of the form $\mathbf{CS}(\mathbf{H})$ where $\mathbf{H}$ is a locally extensible pseudovariety of groups.

Equalities of pseudovarieties $\mathbf{P}(\mathbf{CS}(\mathbf{H})) = \mathbf{J} * \mathbf{CS}(\mathbf{H}) = \mathbf{J} ⓜ \mathbf{CS}(\mathbf{H})$

The expectations pronounced above have actually come true. Indeed, the following statements hold true.

Theorem

For every locally extensible pseudovariety $\mathbf{H}$ *of groups, the equality of pseudovarieties* $\mathbf{P}(\mathbf{CS}(\mathbf{H})) = \mathbf{J} * \mathbf{CS}(\mathbf{H})$ *holds.*

Theorem

For every locally extensible pseudovariety $\mathbf{H}$ *of groups, the equality of pseudovarieties* $\mathbf{J} * \mathbf{CS}(\mathbf{H}) = \mathbf{J} ⓜ \mathbf{CS}(\mathbf{H})$ *holds.*

The latter equalities need not always hold

We complement the previous theorems with the following pieces of knowledge.

Proposition

For every pseudovariety $\mathbf{H}$ *of groups such that the equality* $\mathbf{PH} = \mathbf{J} * \mathbf{H}$ *does not hold, also the equality* $\mathbf{P}(\mathbf{CS}(\mathbf{H})) = \mathbf{J} * \mathbf{CS}(\mathbf{H})$ *does not hold.*

Proposition

For every pseudovariety $\mathbf{H}$ *of groups such that the equality* $\mathbf{J} * \mathbf{H} = \mathbf{J} \textcircled{m} \mathbf{H}$ *does not hold, also the equality* $\mathbf{J} * \mathbf{CS}(\mathbf{H}) = \mathbf{J} \textcircled{m} \mathbf{CS}(\mathbf{H})$ *does not hold.*

Thus the previous theorems need not hold without any additional assumption on the pseudovariety $\mathbf{H}$ of groups.