The wreath product in the automorphism groups of graphs

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SandGAL 2019, Cremona, June 10-13

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Composition of graphs

Definition

Given $G_1 = (X, E_1), \; G_2 = (Y, E_2)$;

Then $G_1 \circ G_2 = (X \times Y, E)$, where E is given by:

- if $\{x_1, x_2\} \in E_1$, then $\{(x_1, y_1), (x_2, y_2)\} \in E$ for all $y_1, y_2 \in Y$;
- if $\{y_1, y_2\} \in E_2$, then $\{(x, y_1), (x, y_2)\} \in E$ for all $x \in X$

Composition of graphs

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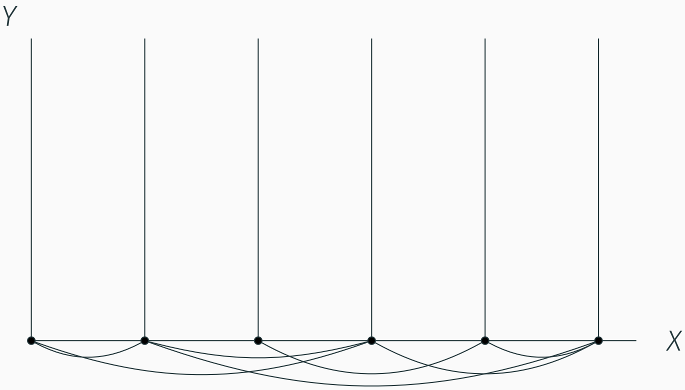

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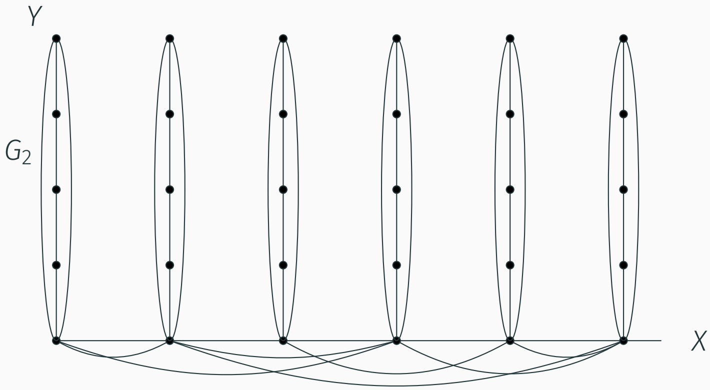

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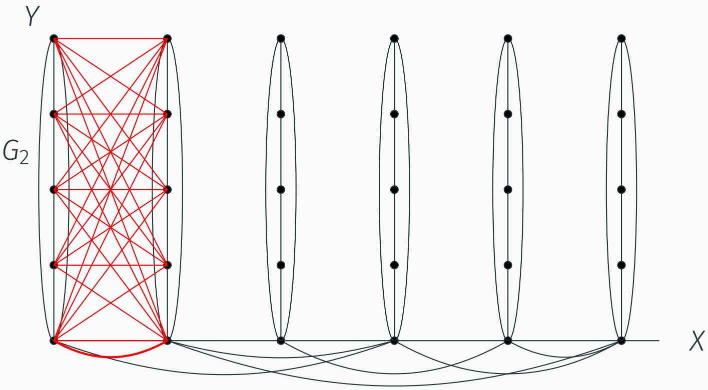

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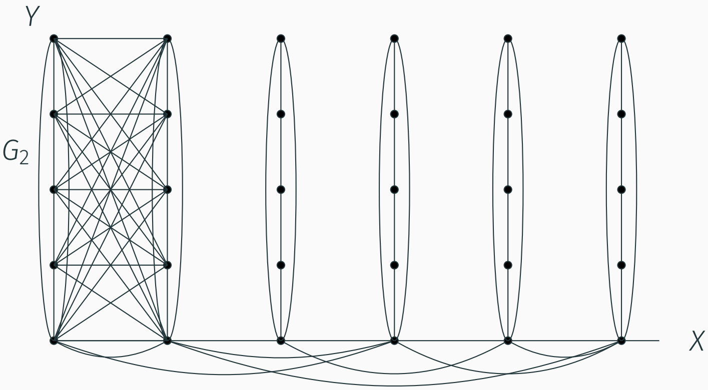

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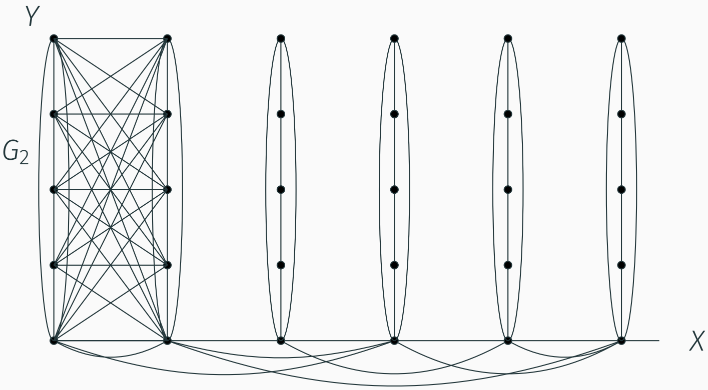
Automorphisms ?



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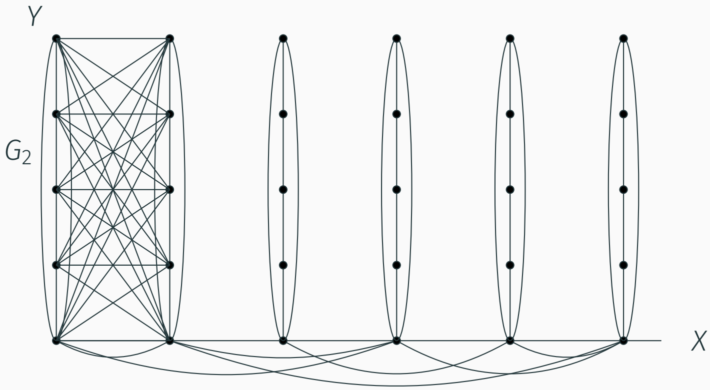

$G_1 = (X, E_1)$

$Aut(G) \supseteq Aut(G_1) \wr Aut(G_2)$

Unnatural isomorphisms?

$Aut(G1 \circ G_2) \supseteq Aut(G_1) \wr Aut(G_2)$

Problem (F. Harary)

When does the equality holds? When has $G1 \circ G_2$ no other, "unnatural" isomorphisms?

Solution: Sabidussi, Hemminger (Duke Math. J. 1961, 1966)

Dobson, Morris (for colored (di)graphs, 2009) – for finite:

$Aut(G_1 \circ G_2) = Aut(G_1) \wr Aut(G_2)$ iff for every color k, the following implication holds:

if G_2 has a pair of ***k*-twins** (two vertices joined by an edge of color k that have exactly the same neighbors in every color), then the ***k*-complement** of G_1 is **connected**.

Converse problem

(Grech, Jeż, AK, 2008): Even if the condition above is not satisfied one can construct a graph G such that $Aut(G) = Aut(G_1) \wr Aut(G_2)$.

- We consider the converse

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PROBLEM

If $Aut(G) = A \wr B$, does it mean that $A = Aur(G_1), B = Aut(G_2$ for some graphs G_1, G_2, and $G = G_1 \circ G_2$?

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- If G is vertex transitive, then YES.

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  $Aut \leftrightarrow G^*$ – orbital graph (colored),
  Galois connection $Perm \leftrightarrow ColGr$

Wreath product

GR – automorphism groups of colored graphs

DGR – automorphism groups of colored digraphs

Theorem

Let A and B be permutation groups. Then, $A \wr B \in GR$ iff $B \in GR \cup \{I_2\}$ and one of the following holds:

1. $A \in GR \cup \{I_2\}$, *or*
2. $A \in DGR \setminus (GR \cup \{I_2\})$ *and B is intransitive.*

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Theorem

Let A and B be permutation groups. Then $A \wr B \in DGR$ iff both $A, B \in DGR$.

Graph $G = G^*(A \wr B)$

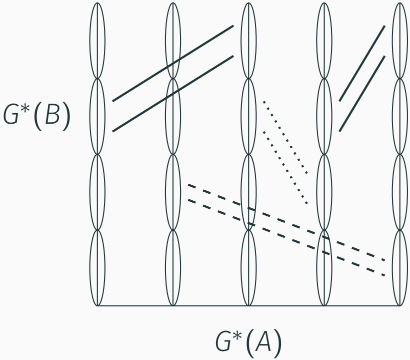

Figure 1: Construction of the orbital graph $G = G^*(A \wr B)$

Product action of the wreath product

$(\beta, (\alpha_x)_{x \in X})$, where $\beta \in Sym(X)$ and $\alpha_x \in Sym(Y)$

---

the **imprimitive action** permutations $\gamma \in Sym(X \times Y)$ given by:

$$(v, w)\gamma = (v\alpha_w, w\beta) \tag{1}$$

for every $(v, w) \in V \times W$.

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the **product action** permutations $\phi \in Sym(Y^X)$ given by:

$$(f\phi)(x) = (f(x\beta))\alpha_x, \tag{2}$$

for any $f \in Y^X$ and any $x \in X$.

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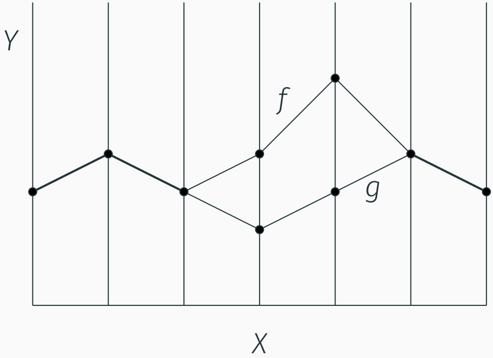

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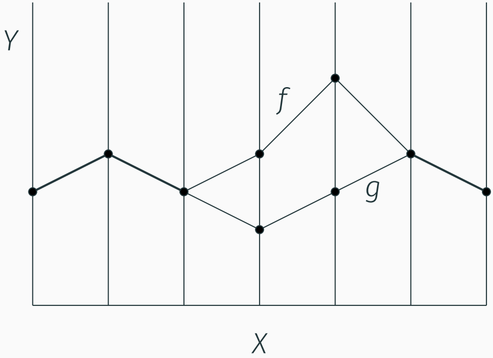

if $|X| = n$ finite, then Y^X – n-tuples $\langle y_1, y_2, \ldots, y_n \rangle$ permuted by $(\alpha_1, \ldots, \alpha_n)$ and $\beta \in S_n$

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orbital graph $G^*(A \wr B)$ – structure **???**

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For any permutation groups A, B, $\quad \overline{A \wr B} \subseteq \overline{A} \wr \overline{B}$.

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Theorem (Grech, Jeż, AK) 2008

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Proposition (for $B = Sym(Y)$**)**

The product $A \wr Sym(Y) \in GR$ if and only if $A \in BGR$.

BGR – orbit-closed permutation groups (action on **subsets**)

Product action of the wreath product: $A \in BGR$

Theorem

*If a permutation group $A \in$ **BGR**, then for any permutation group B, the product $A \wr B \in$ **GR** iff $B \in$ **DGR** and one of the following holds:*

- *B has not transposable orbitals, or*
- *$B \in GR \cup \{I_2\}$.*

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*If a permutation group $A \in$ **BGR**, then for any permutation group A, the product $A \wr B \in DGR$ iff $A \in DGR$.*

Product action of the wreath product: $A \notin BGR$

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Theorem

Let (A, X) and (B, Y) be permutation groups. If $(A, X) \notin BGR$, then the following hold:

1. *If $B \notin DGR^+$, then $A \wr B \notin GR$;*
2. *If $B \in DGR^+$, and either $rank(B) \geq |X| + 1$ or $rank(B) = |X|$ and all orbitals of B are self-paired, then $A \wr B \in GR$.*

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Main open Problem

- *BGR* may be partitioned into small classes

The *symmetry group* of a Boolean function $f : \{0,1\}^n \to \{0,1\}$ set of all permutations σ such that

$$f(x_{1\sigma}, x_{2\sigma}, \ldots, x_{x\sigma}) = f(x_1, x_2, \ldots, x_n)$$

for all $x_1, x_2, \ldots, x_n \in \{0,1\}$.

BGR(2) – class of the symmetry groups of Boolean functions

BGR(k) – class of the symmetry groups of k-valued Boolean functions

- *BGR*$= \bigcup BGR(k)$

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PROBLEM

Is $BGR(k)$ a real hierarchy (unknown permutation groups), or there are only a few exceptions in $BGR \setminus BGR(2)$?