Profinite congruences

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Completeness theorem for equational logic

- Jorge Almeida's talk in Lisbon 2018 was based on the paper *Towards a pseudoequational proof theory* (Portugal. Math., 75 (2), 2018).
- The basic aim was a proof scheme for pseudoidentities.
- Motivation: the completeness theorem for equational logic states that, for a given signature Λ and a set of identities Σ, every identity valid in the variety $[\Sigma]$ admits a proof from Σ.
- Notice that
  - proofs are finite;
  - consequences in proofs are given by substitutions in any contexts (first of all) and (then) transitivity;
  - the set of all identities valid in $[\Sigma]$ is the fully invariant congruence generated by Σ on the free algebra $T_\Lambda(X)$ over a countable set of variables X.

Pseudoequational proof theory

For pseudoidentities, the situation is rather different.

- Recall that a pseudoidentity over a given pseudovariety $\mathbf{U}$ of finite algebras is a formal equality $u = v$ between two elements u and v of some $\mathbf{U}$-free profinite algebra $\overline{\Omega}_A \mathbf{U}$, where A is a finite set.
- We consider a proof scheme where consequences are given by substitutions in any contexts, transitivity and by limits of convergent sequences of proved pseudoidentities.
- Notice that it is again possible to move all substitution steps to the beginning of the proof.

Transfinite proof scheme

- So, for a proof of a particular pseudoidentity, one may stick to the variables which appear in it, so that one ends up working inside $\overline{\Omega}_A\mathbf{U}$ for a given fixed finite A. We work with relations on $\overline{\Omega}_A\mathbf{U}$.
- It also allows us to consider an initial step where we take all consequences of Σ by substitutions (in all possible contexts) and then alternate transitive closure and topological closure.
- The proof scheme is no longer finite.
- We use transfinite induction, with closure steps for successor ordinals and union for limit ordinals. In this way, we construct equivalence relations Σ_α on $\overline{\Omega}_A\mathbf{U}$.
- We put $\tilde{\Sigma} = \Sigma_\alpha$ for α such that $\Sigma_\alpha = \Sigma_{\alpha+2}$.

Completeness conjecture – 2018 version

- Let us write:
  - $\Sigma \models u = v$ to mean that the pseudoidentity $u = v$ is valid in the pseudovariety $[\![\Sigma]\!]$;
  - $\Sigma \vdash u = v$ if $u = v$ can be proved from Σ.
- It is obvious that the enriched proof scheme is sound in the sense that

$$\Sigma \vdash u = v \implies \Sigma \models u = v.$$

Conjecture

For every pseudovariety $\mathbf{U}$ of a finite algebraic type consisting of finitary operations, the enriched proof scheme is complete in the sense that, for every set $\Sigma \cup \{u = v\}$ of $\mathbf{U}$-pseudoidentities,

$$\Sigma \models u = v \implies \Sigma \vdash u = v.$$

Some results from the Lisbon conference

Theorem

The conjecture holds for the pseudovariety of all finite groups.

That is $\mathbf{U} = \mathbf{G}$. Proof is based on properties of groups.

Theorem

For $\mathbf{U} = \mathbf{M}$, the set $\Sigma = \{x^\omega = 1\}$ satisfies the conjecture.

The proof uses deep results concerning κ-reducibility of $\mathbf{G}$.
We also prove the conjecture for usual bases Σ of pseudovarieties $\mathbf{A}, \mathbf{J}, \mathbf{Com}, \mathbf{R}, \mathbf{DA}, \mathbf{CR}$.

Corollary

For $\mathbf{U} = \mathbf{M}$, every set Σ of pseudoidentities defining a group pseudovariety satisfies the conjecture.

Closed congruences

- $\tilde{\Sigma}$ is a closed congruence on $\overline{\Omega}_A\mathbf{U}$ generated by Σ_0, or a fully invariant closed congruence generated by Σ.
  (A congruence is closed if it is a closed subset of the space $\overline{\Omega}_A\mathbf{U} \times \overline{\Omega}_A\mathbf{U}$.)
- The conjecture is equivalent to stating that $\tilde{\Sigma}$ is a profinite congruence in the sense that $\overline{\Omega}_A\mathbf{U}/\tilde{\Sigma}$ is a profinite algebra.
- Every closed congruence on a profinite group is profinite.
- This is not true for profinite semigroups.

What we can say about quotient of profinite semigroups (or algebras) by closed congruences in general?
(We ignore now that $\tilde{\Sigma}$ is fully invariant.)

General observations for profinite algebras

- A topological algebra is profinite if it is a residually finite 0-dimensional compact Hausdorff space with basic operations being continous functions.
- A compact Hausdorff space is 0-dimensional iff it is totally disconnected – such a space is called a Stone space.
- The quotient space X/θ of a compact Hausdorff space X by a closed equivalence relation θ is also compact Hausdorff.

Theorem (Gehrke 2016)

Let A be a profinite algebra and let θ be a closed congruence on A such that A/θ is 0-dimensional. Then A/θ is profinite.

So, the crucial property is 0-dimensionality.

Normal spaces

- A topological space is said to be normal if disjoint closed sets may be separated by disjoint open sets.
- It is an exercise in compactness to show that compact Hausdorff spaces are normal.
- (Urysohn) In a normal space, closed subsets may be separated by continuous functions into the interval $[0, 1]$ ($f : X \to [0, 1]$ such that $f(C) = \{0\}$ and $f(D) = \{1\}$).
- Since the image of a connected space by a continuous function is connected and the connected subspaces of a real interval are the subintervals, we deduce that a countable normal connected space is a singleton.
- Thus a countable normal space is totally disconnected.

Countable Stone spaces

Proposition

The quotient S/θ of a Stone space S by a closed equivalence relation θ of countable index is also a Stone space.

Theorem

Every closed congruence of countable index on a profinite algebra is a profinite congruence.

Corollary

The conjecture holds for the pseudovariety $\mathbf{J}$.

Unary algebras

- Consider a finite set A whose elements are viewed as symbols of unary operations.
- An A-unary algebra is a set U together with a function $a_U : U \to U$ for each $a \in A$.
- These functions generate a subsemigroup S_U of the full transformation semigroup U^U.
- For every semigroup S containing a set A we can define $\mathcal{U}(S)$ as an A-unary algebra with domain S^1 and operations given by $a_{\mathcal{U}(S)}(s) = s \cdot a$.
- The free A-unary algebra over the singleton set $\{x\}$ is isomorphic to $\mathcal{U}(A^+)$.

Profinite unary algebras

- In the profinite case one can prove the analog result: The free profinite A-unary algebra $\overline{\Omega}_{\{x\}}\mathbf{Un}$ over the singleton set $\{x\}$ is isomorphic to $\mathcal{U}(\overline{\Omega}_A\mathbf{S})$, where $\overline{\Omega}_A\mathbf{S}$ is the free profinite semigroup over A.
- Moreover, fully invariant closed congruences of $\overline{\Omega}_{\{x\}}\mathbf{Un}$ correspond to closed congruences of $\overline{\Omega}_A\mathbf{S}$.
- However, we know an example of closed congruence of $\overline{\Omega}_A\mathbf{S}$ which is not profinite.
- This gives a counterexample to the conjecture concerning completeness of pseudoequational proof theory.

Theorem

There is a pseudovariety $\mathbf{U}$ of unary algebras and a closed fully invariant congruence θ on $\overline{\Omega}_{\{x\}}\mathbf{U}$ that is not profinite.

The counterexample

- For $A = \{a, b\}$ we consider $\overline{\Omega}_A\mathbf{K}$.
- Note that the infinite words form a closed subspace of $\overline{\Omega}_A\mathbf{K}$ homeomorphic with the Cantor set.
- Let θ be the equivalence relation on $\overline{\Omega}_A\mathbf{K}$ which identifies two infinite words if and only if they are equal or constitute a pair of the form (wab^ω, wba^ω), with w a finite word; finite words form singleton classes.
- θ is a closed congruence on $\overline{\Omega}_A\mathbf{K}$ and we consider the associated fully invariant closed congruence $\tilde{\theta}$ on $\overline{\Omega}_{\{x\}}\mathbf{K}^u$.
- Then $\overline{\Omega}_{\{x\}}\mathbf{K}^u/\tilde{\theta}$ is not profinite (it contains a closed subspace homeomorphic with the interval $[0, 1]$, which is not 0-dimensional).

Completeness conjecture – 2019 version

Conjecture

For every pseudovariety of finite semigroups, the enriched proof scheme is complete.

Does **K** satisfy the conjecture?

Thank you