Ordered Kovács-Newman semigroups

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- subdirectly indecomposable
- tool for irreducibility results

Irreducibility

Join

$\mathbf{V} \vee \mathbf{W} = \mathbb{HSP}_{fin}(\mathbf{V} \cup \mathbf{W}) = \mathbb{HS}(\{S \times T \mid S \in \mathbf{V}, T \in \mathbf{W}\})$

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Irreducibility

We say that pseudovariety $\mathbf{U}$ is **finite join irreducible** (**fji**) if

$$\mathbf{U} \leq \mathbf{V} \vee \mathbf{W} \text{ then either } \mathbf{U} \leq \mathbf{V} \text{ or } \mathbf{U} \leq \mathbf{W}.$$

Definitions

Preliminary definitions (group context):

- G is **subdirect product** of G_1, G_2 ($G \ll G_1 \times G_2$) if $G \leq G_1 \times G_2$ and projections $\pi_i \colon G \to G_i$ are surjective

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Note that subdirectly indecomposable $\Leftrightarrow$ monolithic.

Definition

Named after Kovács, Newman (1966) studying varieties of groups.

Kovács-Newman (KN) group (Rhodes, Steinberg, 2004)

Non-trivial finite group G is KN **if** whenever there is a diagram

$$G \overset{\varphi}{\twoheadleftarrow} H \ll G_1 \times G_2,$$

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Since $G \in \mathbf{H_1} \vee \mathbf{H_2}$ if there is such diagram with $G_i \in \mathbf{H_i}$, then also the pseudovariety $\mathbb{HSP}_{fin}(G)$, with G being KN group, is a **fji** pseudovariety.

Classification

Theorem (Rhodes, Steinberg, 2009)

Finite group G is KN $\Leftrightarrow$ G is monolithic with non-abelian monolith.

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Example

Group $\mathbb{S}_n$ is KN for $n \geq 5$, as $\mathbb{A}_n$ forms the monolith.

Pseudovariety **G** of finite groups is irreducible.

Semigroup Setting

Definition is analogic to that of KN group:

Kovács-Newman (KN) semigroup

Non-trivial finite semigroup S is KN **if** whenever there is a diagram

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The classification is more complicated. We need to define group mapping semigroups.

Semigroup Setting

Kernel of a semigroup S is a minimal two-sided ideal denoted $K(S)$.

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Semigroup S is GM over its kernel $K(S)$ if S acts faithfully on both left and right of $K(S)$ and $K(S)$ contains non-trivial subgroup.

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Theorem (Rhodes, Steinberg, 2009)

Finite semigroup is a Kovács-Newman semigroup if and only if it is group mapping semigroup over its kernel and the maximal subgroup of the kernel is a KN group (i.e. monolithic with non-abelian monolith).

Applications

Non-isomorphic KN semigroups generate irreducible distinct pseudovarieties.

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Theorem (Rhodes, Steinberg, 2009)

Pseudovariety **CS** of completely simple semigroups is finite join irreducible.

Other **fji** results: **CR**, **DS** ...

Ordered Case

Does this yield some results for the lattice $\mathcal{L}_o(\mathbf{S})$?

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Theorem (JK, 2018)

KN ordered semigroups are precisely KN semigroups ordered by equality.

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Irreducibility of certain pseudovarieties in $\mathcal{L}_o(\mathbf{S})$ studied ($\mathbf{A}, \overline{\mathbf{H}}, \mathbf{B}, \ldots$ by Almeida, Klíma (2015, 2018) using different techniques).

Pseudovariety **CS** in $\mathcal{L}_o(\mathbf{S})$

Corollary (JK, 2018)

Pseudovariety **CS** of ordered completely simple semigroups is finite join irreducible in the lattice $\mathcal{L}_o(\mathbf{S})$.

Closure

Thank you so much for your attention.