A certain lattice of quasivarieties

Tim Koussas
La Trobe University, Melbourne

SandGAL 2019

We assume familiarity with the notions of variety and quasivariety.

Notation

Let $\mathcal{K}$ be a class of semigroups.

We assume familiarity with the notions of variety and quasivariety.

Notation

Let $\mathcal{K}$ be a class of semigroups.

- $\mathbb{V}(\mathcal{K})$ is the variety generated by $\mathcal{K}$.

We assume familiarity with the notions of variety and quasivariety.

Notation

Let $\mathcal{K}$ be a class of semigroups.

- $\mathbb{V}(\mathcal{K})$ is the variety generated by $\mathcal{K}$.
- $\mathbb{Q}(\mathcal{K})$ is the quasivariety generated by $\mathcal{K}$.

We assume familiarity with the notions of variety and quasivariety.

Notation

Let $\mathcal{K}$ be a class of semigroups.

- $\mathbb{V}(\mathcal{K})$ is the variety generated by $\mathcal{K}$.
- $\mathbb{Q}(\mathcal{K})$ is the quasivariety generated by $\mathcal{K}$.

Definition

A quasivariety is called locally finite if all of its finitely generated members are finite.

We assume familiarity with the notions of variety and quasivariety.

Notation

Let $\mathcal{K}$ be a class of semigroups.

- $\mathbb{V}(\mathcal{K})$ is the variety generated by $\mathcal{K}$.
- $\mathbb{Q}(\mathcal{K})$ is the quasivariety generated by $\mathcal{K}$.

Definition

A quasivariety is called locally finite if all of its finitely generated members are finite.

In the early 1980s, Mark Sapir determined which locally finite varieties have a finite number of subquasivarieties. This talk grew out of an analysis of parts of his proof.

Definition

A finite semigroup is called quasicritical if it is **not** in the quasivariety generated by its proper subsemigroups.

Examples:

Definition

A finite semigroup is called quasicritical if it is **not** in the quasivariety generated by its proper subsemigroups.

Examples:

- Any 2-element semigroup is quasicritical.

Definition

A finite semigroup is called quasicritical if it is **not** in the quasivariety generated by its proper subsemigroups.

Examples:

- Any 2-element semigroup is quasicritical.
- A finite Abelian group is quasicritical if and only if it is a cyclic group of prime-power order.

Definition

A finite semigroup is called quasicritical if it is **not** in the quasivariety generated by its proper subsemigroups.

Examples:

- Any 2-element semigroup is quasicritical.
- A finite Abelian group is quasicritical if and only if it is a cyclic group of prime-power order.
- The only quasicritical semilattice is the 2-element semilattice $\mathbf{I}$.

Notation

Let $\mathcal{Q}$ be a quasivariety. The lattice of subquasivarieties of $\mathcal{Q}$ is denoted by $\mathsf{L_q}(\mathcal{Q})$.

Notation

Let $\mathcal{Q}$ be a quasivariety. The lattice of subquasivarieties of $\mathcal{Q}$ is denoted by $\mathsf{L}_\mathsf{q}(\mathcal{Q})$.

Theorem

Let $\mathcal{Q}$ be a locally finite quasivariety, and let $\mathcal{K} \in \mathsf{L}_\mathsf{q}(\mathcal{Q})$. Then $\mathcal{K} = \mathbb{Q}(U)$ for some set U of quasicriticals in $\mathcal{Q}$.

Notation

Let $\mathcal{Q}$ be a quasivariety. The lattice of subquasivarieties of $\mathcal{Q}$ is denoted by $\mathsf{L_q}(\mathcal{Q})$.

Theorem

Let $\mathcal{Q}$ be a locally finite quasivariety, and let $\mathcal{K} \in \mathsf{L_q}(\mathcal{Q})$. Then $\mathcal{K} = \mathbb{Q}(U)$ for some set U of quasicriticals in $\mathcal{Q}$.

In other words, every subquasivariety of $\mathcal{Q}$ is generated by a set of quasicriticals.

Notation

Let $\mathcal{Q}$ be a quasivariety. The lattice of subquasivarieties of $\mathcal{Q}$ is denoted by $\mathsf{L_q}(\mathcal{Q})$.

Theorem

Let $\mathcal{Q}$ be a locally finite quasivariety, and let $\mathcal{K} \in \mathsf{L_q}(\mathcal{Q})$. Then $\mathcal{K} = \mathbb{Q}(U)$ for some set U of quasicriticals in $\mathcal{Q}$.

In other words, every subquasivariety of $\mathcal{Q}$ is generated by a set of quasicriticals.

Theorem

If $\mathcal{Q}$ is a locally finite quasivariety, then $\mathsf{L_q}(\mathcal{Q})$ is finite if and only if $\mathcal{Q}$ contains finitely many quasicriticals up to isomorphism.

$\cdot$	a	b
a	a	b
b	a	b

R

$\cdot$	e	u	0
e	e	u	0
u	0	0	0
0	0	0	0

P

$\cdot$	a	b
a	a	b
b	a	b

R

$\cdot$	e	u	0
e	e	u	0
u	0	0	0
0	0	0	0

P

The dual semigroups of **R** and **P** are denoted **L** and **Q**, respectively.

$\cdot$	a	b
a	a	b
b	a	b

R

$\cdot$	e	u	0
e	e	u	0
u	0	0	0
0	0	0	0

P

The dual semigroups of **R** and **P** are denoted **L** and **Q**, respectively.

We also write **I** for the 2-element semilattice, and **N** for the 2-element null semigroup.

Definition

A semigroup is called aperiodic if all of its subgroups are trivial.

Definition

A semigroup is called aperiodic if all of its subgroups are trivial.

Conjecture (my motivation)

Let $\mathbf{S}$ *be a finite aperiodic semigroup. The following are equivalent:*

(i) $\mathbf{S}$ *dualisable;*

(ii) $\mathbf{S} \in \mathbb{V}(\mathbf{L}, \mathbf{R}, \mathbf{I}, \mathbf{N}) \cup \mathbb{V}(\mathbf{R}, \mathbf{P}) \cup \mathbb{V}(\mathbf{L}, \mathbf{Q})$.

Definition

A semigroup is called aperiodic if all of its subgroups are trivial.

Conjecture (my motivation)

Let $\mathbf{S}$ *be a finite aperiodic semigroup. The following are equivalent:*

(i) $\mathbf{S}$ *dualisable;*

(ii) $\mathbf{S} \in \mathbb{V}(\mathbf{L}, \mathbf{R}, \mathbf{I}, \mathbf{N}) \cup \mathbb{V}(\mathbf{R}, \mathbf{P}) \cup \mathbb{V}(\mathbf{L}, \mathbf{Q})$.

So far, I have completed the proof of (ii) $\implies$ (i), with one more case needed for the converse.

Definition

A semigroup is called aperiodic if all of its subgroups are trivial.

Conjecture (my motivation)

Let $\mathbf{S}$ *be a finite aperiodic semigroup. The following are equivalent:*

(i) $\mathbf{S}$ *dualisable;*

(ii) $\mathbf{S} \in \mathbb{V}(\mathbf{L}, \mathbf{R}, \mathbf{I}, \mathbf{N}) \cup \mathbb{V}(\mathbf{R}, \mathbf{P}) \cup \mathbb{V}(\mathbf{L}, \mathbf{Q})$.

So far, I have completed the proof of (ii) $\implies$ (i), with one more case needed for the converse. A key component of the proof of (ii) $\implies$ (i) was an explicit description of the subquasivariety lattices of $\mathbb{V}(\mathbf{L}, \mathbf{R}, \mathbf{I}, \mathbf{N})$ and $\mathbb{V}(\mathbf{R}, \mathbf{P})$.

The description of $\mathsf{L}_{\mathbf{q}}(\mathbb{V}(\mathbf{L}, \mathbf{R}, \mathbf{I}, \mathbf{N}))$ turned out to be straightforward. But $\mathsf{L}_{\mathbf{q}}(\mathbb{V}(\mathbf{R}, \mathbf{P}))$ was unexpectedly interesting; this will be the focus of our talk.

The description of $\mathrm{L}_{\mathbf{q}}(\mathbb{V}(\mathbf{L},\mathbf{R},\mathbf{I},\mathbf{N}))$ turned out to be straightforward. But $\mathrm{L}_{\mathbf{q}}(\mathbb{V}(\mathbf{R},\mathbf{P}))$ was unexpectedly interesting; this will be the focus of our talk.

Because $\mathbb{V}(\mathbf{R},\mathbf{P})$ is locally finite, we can find all of its subquasivarieties by finding its quasicriticals (by the earlier result).

The description of $\mathsf{L_q}(\mathbb{V}(\mathbf{L}, \mathbf{R}, \mathbf{I}, \mathbf{N}))$ turned out to be straightforward. But $\mathsf{L_q}(\mathbb{V}(\mathbf{R}, \mathbf{P}))$ was unexpectedly interesting; this will be the focus of our talk.

Because $\mathbb{V}(\mathbf{R}, \mathbf{P})$ is locally finite, we can find all of its subquasivarieties by finding its quasicriticals (by the earlier result).

As part of Mark Sapir's work (mentioned earlier), he described the quasicriticals in some varieties, one of which was $\mathbb{V}(\mathbf{R}, \mathbf{P})$.

The description of $\mathsf{L_q}(\mathbb{V}(\mathbf{L}, \mathbf{R}, \mathbf{I}, \mathbf{N}))$ turned out to be straightforward. But $\mathsf{L_q}(\mathbb{V}(\mathbf{R}, \mathbf{P}))$ was unexpectedly interesting; this will be the focus of our talk.

Because $\mathbb{V}(\mathbf{R}, \mathbf{P})$ is locally finite, we can find all of its subquasivarieties by finding its quasicriticals (by the earlier result).

As part of Mark Sapir's work (mentioned earlier), he described the quasicriticals in some varieties, one of which was $\mathbb{V}(\mathbf{R}, \mathbf{P})$.

Lemma (Sapir)

Up to isomorphism, the only quasicriticals in $\mathbb{V}(\mathbf{R}, \mathbf{P})$ *are* $\mathbf{R}$, $\mathbf{R}^0$, $\mathbf{I}$, $\mathbf{N}$, *and* $\mathbf{P}$.

Definition

Let **A** be a (partially) ordered set. A down-set of **A** is a subset U of A that is decreasing; that is, $y \leqslant x \in U \implies y \in U$.

Definition

Let $\mathbf{A}$ be a (partially) ordered set. A down-set of $\mathbf{A}$ is a subset U of A that is decreasing; that is, $y \leqslant x \in U \implies y \in U$.

- We could find all subquasivarieties of $\mathbb{V}(\mathbf{R}, \mathbf{P})$ by forming $\mathbb{Q}(U)$ for each $U \subseteq \{\mathbf{R}, \mathbf{R}^0, \mathbf{I}, \mathbf{N}, \mathbf{P}\}$.

Definition

Let $\mathbf{A}$ be a (partially) ordered set. A down-set of $\mathbf{A}$ is a subset U of A that is decreasing; that is, $y \leqslant x \in U \implies y \in U$.

- We could find all subquasivarieties of $\mathbb{V}(\mathbf{R}, \mathbf{P})$ by forming $\mathbb{Q}(U)$ for each $U \subseteq \{\mathbf{R}, \mathbf{R}^0, \mathbf{I}, \mathbf{N}, \mathbf{P}\}$.
- But there are 32 choices for U. We can be more efficient.

Definition

Let $\mathbf{A}$ be a (partially) ordered set. A down-set of $\mathbf{A}$ is a subset U of A that is decreasing; that is, $y \leqslant x \in U \implies y \in U$.

- We could find all subquasivarieties of $\mathbb{V}(\mathbf{R}, \mathbf{P})$ by forming $\mathbb{Q}(U)$ for each $U \subseteq \{\mathbf{R}, \mathbf{R}^0, \mathbf{I}, \mathbf{N}, \mathbf{P}\}$.
- But there are 32 choices for U. We can be more efficient.
- Let us order the quasicriticals $\{\mathbf{R}, \mathbf{R}^0, \mathbf{I}, \mathbf{N}, \mathbf{P}\}$ by embedding (i.e., $\mathbf{S} \leqslant \mathbf{T}$ if $\mathbf{S}$ embeds into $\mathbf{T}$).

Definition

Let $\mathbf{A}$ be a (partially) ordered set. A down-set of $\mathbf{A}$ is a subset U of A that is decreasing; that is, $y \leqslant x \in U \implies y \in U$.

- We could find all subquasivarieties of $\mathbb{V}(\mathbf{R}, \mathbf{P})$ by forming $\mathbb{Q}(U)$ for each $U \subseteq \{\mathbf{R}, \mathbf{R}^0, \mathbf{I}, \mathbf{N}, \mathbf{P}\}$.
- But there are 32 choices for U. We can be more efficient.
- Let us order the quasicriticals $\{\mathbf{R}, \mathbf{R}^0, \mathbf{I}, \mathbf{N}, \mathbf{P}\}$ by embedding (i.e., $\mathbf{S} \leqslant \mathbf{T}$ if $\mathbf{S}$ embeds into $\mathbf{T}$).
- Then every subquasivariety is generated by a down-set of this ordered set.

Definition

Let $\mathbf{A}$ be a (partially) ordered set. A down-set of $\mathbf{A}$ is a subset U of A that is decreasing; that is, $y \leqslant x \in U \implies y \in U$.

- We could find all subquasivarieties of $\mathbb{V}(\mathbf{R}, \mathbf{P})$ by forming $\mathbb{Q}(U)$ for each $U \subseteq \{\mathbf{R}, \mathbf{R}^0, \mathbf{I}, \mathbf{N}, \mathbf{P}\}$.
- But there are 32 choices for U. We can be more efficient.
- Let us order the quasicriticals $\{\mathbf{R}, \mathbf{R}^0, \mathbf{I}, \mathbf{N}, \mathbf{P}\}$ by embedding (i.e., $\mathbf{S} \leqslant \mathbf{T}$ if $\mathbf{S}$ embeds into $\mathbf{T}$).
- Then every subquasivariety is generated by a down-set of this ordered set.
- This is simply because a quasivariety is closed under forming subsemigroups.

$\cdot$	e	u	0
e	e	u	0
u	0	0	0
0	0	0	0

P

$\cdot$	a	b	0
a	a	b	0
b	a	b	0
0	0	0	0

$\mathbf{R}^0$

$\cdot$	0	1
0	0	0
1	0	1

I

$\cdot$	0	$\not 0$
0	0	0
$\not 0$	0	0

N

$\cdot$	e	u	0
e	e	u	0
u	0	0	0
0	0	0	0

P

$\cdot$	a	b	0
a	a	b	0
b	a	b	0
0	0	0	0

$\mathbf{R}^0$

$\cdot$	0	1
0	0	0
1	0	1

I

$\cdot$	0	$\not{0}$
0	0	0
$\not{0}$	0	0

N

Ordering the quasicriticals by embedding:

$\cdot$	e	u	0
e	e	u	0
u	0	0	0
0	0	0	0

P

$\cdot$	a	b	0
a	a	b	0
b	a	b	0
0	0	0	0

$\mathbf{R}^0$

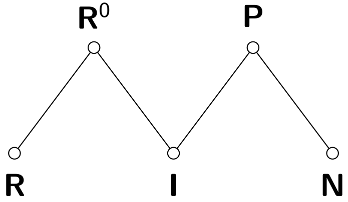
$\cdot$	0	1
0	0	0
1	0	1

I

$\cdot$	0	$\emptyset$
0	0	0
$\emptyset$	0	0

N

Ordering the quasicriticals by embedding:



$\cdot$	e	u	0
e	e	u	0
u	0	0	0
0	0	0	0

$\mathbf{P}$

$\cdot$	a	b	0
a	a	b	0
b	a	b	0
0	0	0	0

$\mathbf{R}^0$

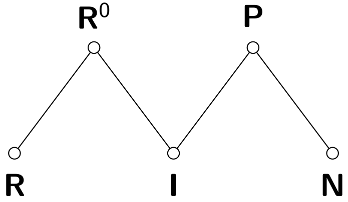
$\cdot$	0	1
0	0	0
1	0	1

$\mathbf{I}$

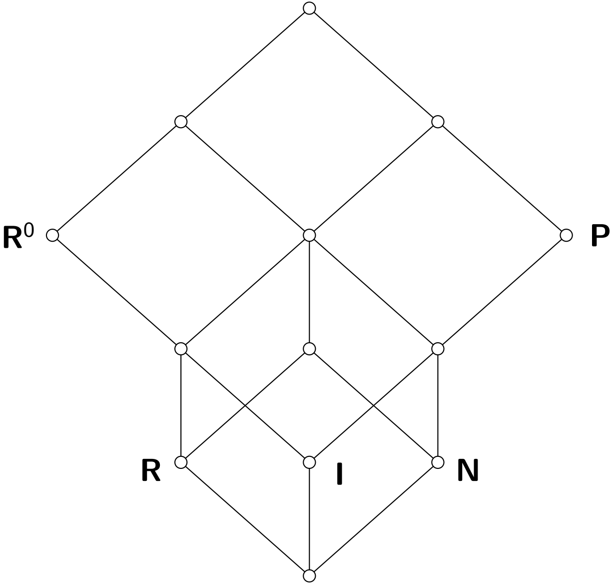
$\cdot$	0	$\emptyset$
0	0	0
$\emptyset$	0	0

$\mathbf{N}$

Ordering the quasicriticals by embedding:



Example: $\mathbb{Q}(\mathbf{R}^0, \mathbf{N})$ may also be generated by $\{\mathbf{R}, \mathbf{R}^0, \mathbf{I}, \mathbf{N}\}$.


R^0
P
R
I
N

Lemma (Sapir)

Up to isomorphism, the only quasicriticals in $\mathbb{V}(\mathbf{R}, \mathbf{P})$ *are* $\mathbf{R}$, $\mathbf{R}^0$, $\mathbf{I}$, $\mathbf{N}$, *and* $\mathbf{P}$.

Lemma (Sapir)

Up to isomorphism, the only quasicriticals in $\mathbb{V}(\mathbf{R}, \mathbf{P})$ *are* $\mathbf{R}$, $\mathbf{R}^0$, $\mathbf{I}$, $\mathbf{N}$, *and* $\mathbf{P}$.

Definition

We denote by **C** the 4-element semigroup below.

$\cdot$	e	u	a	b
e	e	u	a	b
u	a	b	a	b
a	a	b	a	b
b	a	b	a	b

Lemma (Sapir)

Up to isomorphism, the only quasicriticals in $\mathbb{V}(\mathbf{R}, \mathbf{P})$ *are* $\mathbf{R}$, $\mathbf{R}^0$, $\mathbf{I}$, $\mathbf{N}$, *and* $\mathbf{P}$.

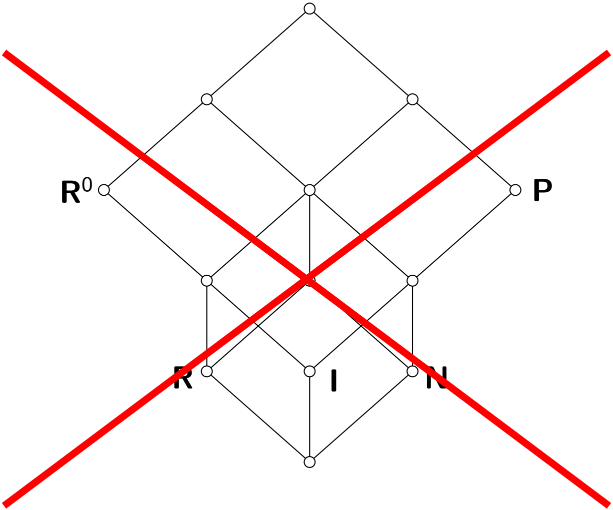
Definition

We denote by **C** the 4-element semigroup below.

$\cdot$	e	u	a	b
e	e	u	a	b
u	a	b	a	b
a	a	b	a	b
b	a	b	a	b

Proposition ($\mathbf{C}$ = 'Counterexample')

$\mathbf{C}$ *is a quasicritical semigroup in* $\mathbb{V}(\mathbf{R}, \mathbf{P})$.



$\mathsf{L_q}(\mathbb{V}(\mathbf{R},\mathbf{P}))$

Well, maybe **C** is the only other quasicritical...

Well, maybe **C** is the only other quasicritical...

Proposition (no, it isn't)

$\mathbf{C}^0$ *is a quasicritical semigroup in* $\mathbb{V}(\mathbf{R}, \mathbf{P})$.

Well, maybe **C** is the only other quasicritical...

Proposition (no, it isn't)

$\mathbf{C}^0$ *is a quasicritical semigroup in* $\mathbb{V}(\mathbf{R}, \mathbf{P})$.

A computer search found no others of size $\leqslant 7$.

Well, maybe **C** is the only other quasicritical...

Proposition (no, it isn't)

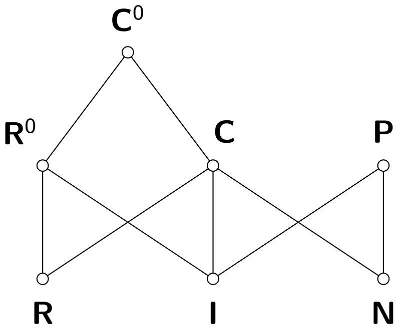
$\mathbf{C}^0$ *is a quasicritical semigroup in* $\mathbb{V}(\mathbf{R}, \mathbf{P})$.

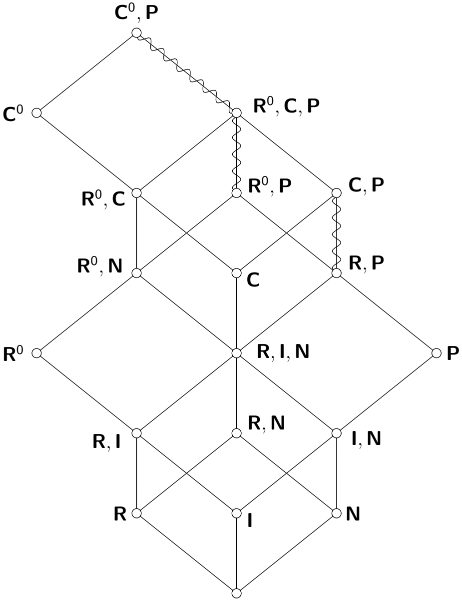
A computer search found no others of size $\leqslant 7$.

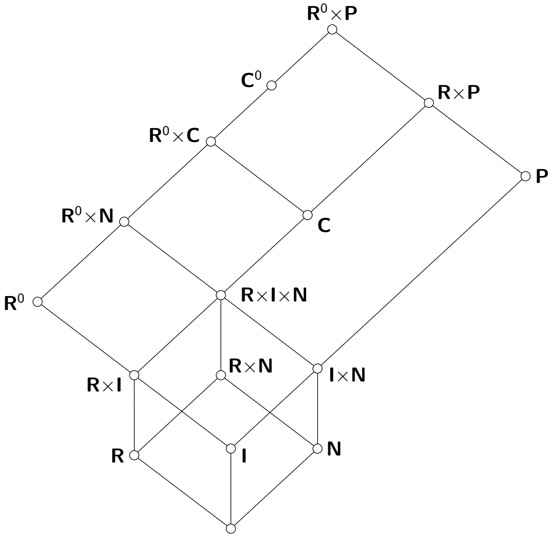
Eventually, we obtained:

Theorem (Koussas)

Up to isomorphism, the only quasicriticals in $\mathbb{V}(\mathbf{R}, \mathbf{P})$ *are* $\mathbf{R}$, $\mathbf{R}^0$, $\mathbf{I}$, $\mathbf{N}$, $\mathbf{P}$, $\mathbf{C}$, *and* $\mathbf{C}^0$.


C0
R0
C
P
R
I
N


$\mathbf{C}^0, \mathbf{P}$
$\mathbf{C}^0$
$\mathbf{R}^0, \mathbf{C}, \mathbf{P}$
$\mathbf{R}^0, \mathbf{C}$
$\mathbf{R}^0, \mathbf{P}$
$\mathbf{C}, \mathbf{P}$
$\mathbf{R}^0, \mathbf{N}$
$\mathbf{C}$
$\mathbf{R}, \mathbf{P}$
$\mathbf{R}^0$
$\mathbf{R}, \mathbf{I}, \mathbf{N}$
$\mathbf{P}$
$\mathbf{R}, \mathbf{N}$
$\mathbf{R}, \mathbf{I}$
$\mathbf{I}, \mathbf{N}$
$\mathbf{R}$
$\mathbf{I}$
$\mathbf{N}$



$\mathsf{L_q}(\mathbb{V}(\mathbf{R},\mathbf{P}))$

- It is not clear if Mark Sapir's main results still hold. The lattice $\mathsf{L}_\mathsf{q}(\mathbb{V}(\mathbf{R}, \mathbf{P}))$ is still finite; it is just bigger than we thought.

- It is not clear if Mark Sapir's main results still hold. The lattice $\mathsf{L_q}(\mathbb{V}(\mathbf{R}, \mathbf{P}))$ is still finite; it is just bigger than we thought.
- To resolve the problem, one would need to extend our description of quasicriticals to the variety $\mathbb{V}(\mathbf{R}, \mathbf{P}, \mathbf{G})$, where $\mathbf{G}$ is an Abelian group.

- It is not clear if Mark Sapir's main results still hold. The lattice $\mathsf{L_q}(\mathbb{V}(\mathbf{R},\mathbf{P}))$ is still finite; it is just bigger than we thought.
- To resolve the problem, one would need to extend our description of quasicriticals to the variety $\mathbb{V}(\mathbf{R},\mathbf{P},\mathbf{G})$, where **G** is an Abelian group.
- The aperiodic case was surprisingly difficult, so the general case involving groups could turn out to be very difficult.

- It is not clear if Mark Sapir's main results still hold. The lattice $\mathsf{L}_\mathsf{q}(\mathbb{V}(\mathbf{R}, \mathbf{P}))$ is still finite; it is just bigger than we thought.
- To resolve the problem, one would need to extend our description of quasicriticals to the variety $\mathbb{V}(\mathbf{R}, \mathbf{P}, \mathbf{G})$, where $\mathbf{G}$ is an Abelian group.
- The aperiodic case was surprisingly difficult, so the general case involving groups could turn out to be very difficult.
- For now, I will (hopefully) publish the results of this talk.

Grazie!