Partial actions and proper extensions of two-sided restriction semigroups

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Partial actions of groups

- G - a group, X - a set
- A left partial action (Exel, 1998) of G on X is a map $Z \subseteq G \times X \to X$, $(g, x) \mapsto g \cdot x$, such that:
  1. $\exists g \cdot x$ implies that $\exists g^{-1} \cdot (g \cdot x)$ and $g^{-1} \cdot (g \cdot x) = x$;
  2. $\exists g \cdot (h \cdot x)$ implies that $\exists gh \cdot x$ and $\exists g \cdot (h \cdot x) = gh \cdot x$;
  3. $\exists 1 \cdot x$ for every $x \in X$ and $1 \cdot x = x$.
- If G acts on Y and $X \subseteq Y$, the action restricts to a partial action of G on X.
- A partial action of G on X is globalizable: it arises as a restriction of an action of G on some $Y \supseteq X$.

Partial actions and premorphisms

- G acts partially on X via $(g, x) \mapsto g \cdot x$
- $g \in G$: $\varphi_g \in \mathcal{I}(X)$ is given by

$$\mathrm{dom}\varphi_g = \{x \in X \colon \exists g \cdot x\}$$

$$\varphi_g(x) = g \cdot x,\ x \in \mathrm{dom}\varphi_g$$

- The map $g \mapsto \varphi_g$ is a premorphism, that is, it satisfies axioms:

(PM1) $\varphi_g \varphi_h \leq \varphi_{gh}$;

(Inv) $\varphi_{g^{-1}} = \varphi_g^{-1}$;

(U) $\varphi(1) = 1$.

- There is a one-to-one correspondence between partial actions of G on X and premorphisms $G \to \mathcal{I}(X)$.
- This extends to partial actions of inverse semigroups and premorphisms between inverse semigroups (the latter known since McAlister and Reilly, 1977).

Partial actions of monoids

- M - a monoid, X - a set
- A left partial action (Megrelishvili and Schröder, 2004) of M on X is a map $Z \subseteq M \times X \to X$, $(m, x) \mapsto m \cdot x$, such that:
  1. If $\exists n \cdot x$ then $\exists m \cdot (n \cdot x)$ if and only if $\exists mn \cdot x$, in which case $m \cdot (n \cdot x) = mn \cdot x$;
  2. $\exists 1 \cdot x$ for every $x \in X$ and $1 \cdot x = x$.
- If M is a group, this is equivalent to the definition of a partial action of a group.
- Partial actions of monoids as defined above are globalizable.

Restriction semigroups

A left restriction semigroup is an algebra $(S; \cdot, ^+)$ of type $(2, 1)$ where $(S; \cdot)$ is a semigroup and:

$$x^+x = x, \;\; x^+y^+ = y^+x^+, \;\; (x^+y)^+ = x^+y^+, \;\; (xy)^+x = xy^+.$$

A right restriction semigroup is an algebra $(S; \cdot, ^*)$ of type $(2, 1)$ where $(S; \cdot)$ is a semigroup and:

$$xx^* = x, \;\; x^*y^* = y^*x^*, \;\; (xy^*)^* = x^*y^*, \;\; y(xy)^* = x^*y.$$

A two-sided restriction semigroup, or just a restriction semigroup, is an algebra $(S; \cdot, ^*, ^+)$, where $(S; \cdot, ^+)$ is a left restriction semigroup, $(S; \cdot, ^*)$ is a right restriction semigroup, and:

$$(x^+)^* = x^+, \;\; (x^*)^+ = x^*.$$

The semilattice of projections of S:

$$P(S) = \{s^* : s \in S\} = \{s^+ : s \in S\}$$

$e \in P(S) \Rightarrow e^2 = e, e^* = e^+ = e$

Some notions and examples

- If $S = P(S)$ then S is a semilattice. Conversely, any semilattice E is a restriction semigroup with $e^* = e^+ = e$.
- An inverse semigroup S is restriction with $s^* = s^{-1}s$ and $s^+ = ss^{-1}$.
- $\mathcal{PT}(X)$ is right restriction with respect to the $f^* = \mathrm{id}_{\mathrm{dom} f}$.
- Any right restriction semigroup is isomorphic to a subsemigroup of some $\mathcal{PT}(X)$ closed under *.
- Partial order: $s \leq t$ if $s = ts^*$
- The compatibility relation: $s \sim t \Leftrightarrow st^* = ts^*$ and $t^+s = s^+t$.
- If $|P(S)| = 1$ then S is a monoid with $P(S) = \{1\}$ called a reduced restriction semigroup. Conversely, any monoid M is a (reduced) restriction semigroup with $m^* = m^+ = 1$ for all $m \in M$.
- σ - the minimum reduced restriction semigroup congruence on S
- S is called proper if $\sigma = \sim$. Proper restriction semigroups generalize E-unitary inverse semigroups.

Partial monoid actions and premorphisms

- A map $M \to S$ (where S - right restriction, two-sided restriction, inverse), $m \mapsto \varphi_m$, is a premorphism, if it satisfies axioms:

(PM1) $\varphi_m \varphi_n \leq \varphi_{mn}$;

(U) $\varphi(1) = 1$.

- It is a right strong premorphism (Hollings, 2007, for S right restriction), if, in addition:

(Sr) $\varphi_m \varphi_n = \varphi_{mn} \varphi_n^*$.

- Premorphisms to $\mathcal{PT}(X)$ satisfying (Sr) correspond to partial actions in the sense of Megrelishvili and Schröder.

Structure of proper restriction semigroups

- M a monoid, X a semilattice
- M acts partially on X by order automorphisms between (non-empty) order ideals of X.
- Equivalently, $\varphi\colon M \to \mathcal{I}(X)$ is a premorphism.
- $M \ltimes_\varphi X$ - the partial action product - is a proper restriction semigroup.
- S - proper restriction semigroup
- $\varphi\colon S/\sigma \to \mathcal{I}(P(S))$, $m \mapsto \varphi_m$, the underlying premorphism of S:

$$\mathrm{dom}\,\varphi_m = \{e \in P(S)\colon e \leq s^*, \text{ for some } s \text{ satisfying } \sigma^\natural(s) = m\}$$

$$\varphi_m(e) = (se)^+, \text{ where } \sigma^\natural(s) = m$$

- $S \simeq P(S) \ltimes_\varphi S/\sigma$ (Cornock and Gould, 2011)
- This extends the Petrich-Reilly result (1979) on the structure of E-unitary inverse semigroups.

Premorphisms between restriction semigroups

- S, T - (two-sided) restriction semigroups
- A map $\varphi\colon S \to T$ will be called a premorphism if the following conditions hold:

(PM1) $\varphi(s)\varphi(t) \leq \varphi(st)$ for all $s, t \in S$;

(PM2) $\varphi(s)^* \leq \varphi(s^*)$ for all $s \in S$;

(PM3) $\varphi(s)^+ \leq \varphi(s^+)$ for all $s \in S$.

- Remark. Let S be a monoid and T a restriction monoid. If $\varphi(S)$ generates T then (PM1), (PM2) and (PM3) are equivalent to (PM1) and (U).
- Premorphisms $S \to \mathcal{I}(X)$ correspond to partial actions of S by partial bijections.

Proper extensions of restriction semigroups

- S, T - (two-sided) restriction semigroups
- A morphism $\psi\colon S \to T$ is called proper if it is surjective and

$$\psi(s) = \psi(t) \Rightarrow s \sim t, \text{ for all } s, t \in S.$$

- Proper morphisms between restriction semigroups generalize:
  - (i) idempotent-pure morphisms between inverse semigroups (S and T inverse);
  - (ii) proper restriction semigroups:

    if T is reduced then $\psi\colon S \to T$ is proper iff S is a proper restriction semigroup, $T \simeq S/\sigma$ and ψ equivalent to $\sigma^{\natural}\colon S \to S/\sigma$.
- For S, T right restriction semigroups, proper extensions have been introduced and studied by Gomes (2005).

A construction

- S - a restriction semigroup, Y - a semilattice, $q\colon Y \to P(S)$ a morphism of semilattices, $\varphi\colon S \to \mathcal{I}(Y)$ a premorphism and:

  (A1) for all $s \in S$: $\mathrm{dom}\,\varphi_s$ and $\mathrm{ran}\,\varphi_s$ are order ideals

  (A2) for all $s \in S$: φ_s is an order automorphism

  (A3) for all $e \in P(S)$: $(q^{-1}(e))^{\downarrow} \subseteq \mathrm{dom}\,\varphi_e \subseteq q^{-1}(e^{\downarrow})$

  (A4) for all $s \in S$: $\mathrm{dom}\,\varphi_s \cap q^{-1}(s^*) \neq \varnothing$

- The partial action product

$$Y \rtimes_{\varphi}^{q} S = \{(y,s) \in Y \times S \colon y \in \mathrm{ran}\,\varphi_s \text{ and } q(y) = s^+\};$$

- The operations $\cdot, ^*, ^+$:

$$(y,s)(x,t) = (\varphi_s(\varphi_s^{-1}(y) \wedge x), st)$$

$$(y,s)^* = (\varphi_s^{-1}(y), s^*),\ (y,s)^+ = (y, s^+)$$

- Proposition. $Y \rtimes_{\varphi}^{q} S$ is a restriction semigroup, and $\Psi\colon Y \rtimes_{\varphi}^{q} S \to S$, $(y,s) \mapsto s$, is a proper morphism.

The two underlying premorphisms of a proper extension

- $\psi\colon T \to S$ - a proper morphism
- For $s \in S$ we define $\widehat{\psi}_s, \widetilde{\psi}_s \in \mathcal{I}(P(T))$:

$$\operatorname{dom}\widehat{\psi}_s = \{e \in P(T)\colon e \leq t^* \text{ for some } t \in T \text{ such that } \psi(t) \leq s\},$$

$$\operatorname{dom}\widetilde{\psi}_s = \{e \in P(T)\colon e \leq t^* \text{ for some } t \in T \text{ such that } \psi(t) = s\}.$$

- For $e \in \operatorname{dom}\widehat{\psi}_s$ (resp. $e \in \operatorname{dom}\widetilde{\psi}_s$) we set

$$\widehat{\psi}_s(e) = (te)^+ \text{ where } t \in T \text{ is such that } e \leq t^* \text{ and } \psi(t) \leq s \text{ (resp. } \psi(t) = s\text{)}.$$

- $\widehat{\psi}$ - the upper underlying premorphism of ψ
- $\widetilde{\psi}$ - the lower underlying premorphism of ψ

Structure of proper extensions

- $\psi\colon T \to S$ - a proper morphism between restriction semigroups
- $p = \psi|_{P(T)}$

Theorem (M. Dokuchaev, M. Khrypchenko, GK, 2019)

$$T \;\simeq\; P(T) \rtimes^{p}_{\widehat{\psi}} S \;\simeq\; P(T) \rtimes^{p}_{\widetilde{\psi}} S$$

- If S is reduced, we obtain the Cornock-Gould structure result on proper restriction semigroups.
- If S, T are inverse semigroups, we obtain a 'partial action' variant of O'Carroll's (1977) result on the structure of idempotent pure extensions: restricting O'Carroll's action, one obtains $\widehat{\psi}$.

Strong and locally strong premorphisms

- S, T - restriction semigroups
- A premorphism $\varphi\colon S \to T$ is called strong if it satisfies:

  (Sr) $\varphi(s)\varphi(t) = \varphi(st)\varphi(t)^*$ for all $s, t \in S$;

  (Sl) $\varphi(s)\varphi(t) = \varphi(s)^+\varphi(st)$ for all $s, t \in S$.
- A premorphism $\varphi\colon S \to T$ is called locally strong if it satisfies:

  (LSr) $\varphi(st^+)\varphi(t) = \varphi(st)\varphi(t)^*$ for all $s, t \in S$;

  (LSl) $\varphi(s)\varphi(s^*t) = \varphi(s)^+\varphi(st)$ for all $s, t \in S$.
- If S is a monoid (or a group) then strong $=$ locally strong.
- Proposition
  - Locally strong premorphisms extend premorphisms between inverse semigroup satisfying $\varphi(s^{-1}) = \varphi(s)^{-1}$, for all $s \in S$.
  - A premorphism $\varphi\colon S \to T$ is strong if and only if it is locally strong and order-preserving.

Strongness and local strongness of $\widehat{\psi}$ and $\widetilde{\psi}$

- $\psi\colon T \to S$ - proper morphism between restriction semigroups
- The premorphism $\widehat{\psi}$ is necessarily order-preserving
- The premorphism $\widetilde{\psi}$ is order-preserving if and only if $\widetilde{\psi} = \widehat{\psi}$.
- Theorem (M. Dokuchaev, M. Khrypchenko, GK, 2019). The following statements are equivalent:

  1. $\widehat{\psi}$ satisfies condition (LSr) (respectively (LSl));
  2. $\widehat{\psi}$ satisfies condition (Sr) (respectively (Sl));
  3. $\widetilde{\psi}$ satisfies condition (LSr) (respectively (LSl)).

  Consequently, $\widehat{\psi}$ is strong if and only if $\widehat{\psi}$ is locally strong if and only if $\widetilde{\psi}$ is locally strong.

Categories of partial actions of S

- The category $\mathcal{A}(S)$:
  - objects: (α, p, X) where X is a semilattice, $p\colon X \to P(S)$ a morphism of semilattices and $\alpha\colon S \to \mathcal{I}(X)$ a premorphism, so that conditions (A1)–(A4) are satisfied.
  - a morphism from (α, p_α, X) to (β, p_β, Y) is a semilattice morphism $p\colon X \to Y$ such that:
    (M1) $p_\alpha = p_\beta p$;
    (M2) $p(\operatorname{dom}\alpha_s) \subseteq \operatorname{dom}\beta_s$ and $\beta_s(p(e)) = p(\alpha_s(e))$ for all $e \in \operatorname{dom}\alpha_s$;
    (M3) $p(\operatorname{dom}\alpha_s \cap p_\alpha^{-1}(s^*)) = \operatorname{dom}\beta_s \cap p_\beta^{-1}(s^*)$ for all $s \in S$.
  - The definition of a morphism agrees with that in the sense of Abadie (2003).
- The subcategories $\widetilde{\mathcal{A}}(S)$ and $\widehat{\mathcal{A}}(S)$ of $\mathcal{A}(S)$ contain as objects premorphisms of the form $\widehat{\psi}$ and $\widetilde{\psi}$, respectively.
- If S is a monoid (in particular, a group), $\widetilde{\mathcal{A}}(S) = \widehat{\mathcal{A}}(S) = \mathcal{A}(S)$.
- Theorem (M. Dokuchaev, M. Khrypchenko, GK, 2019).
  - The category $\widehat{\mathcal{A}}(S)$ is a reflective subcategory of $\mathcal{A}(S)$;
  - $\widetilde{\mathcal{A}}(S)$ is a coreflective subcategory of $\mathcal{A}(S)$;
  - The categories $\widehat{\mathcal{A}}(S)$ and $\widetilde{\mathcal{A}}(S)$ are isomorphic.

Equivalence of categories of partial actions and proper extensions of S

- S - restriction semigroup
- The category $\mathcal{P}(S)$:
  - objects: proper morphisms $\psi\colon T \to S$ where T is a restriction semigroup
  - a morphism from $\psi_1\colon T_1 \to S$ to $\psi_2\colon T_2 \to S$ is a morphism $\gamma\colon T_1 \to T_2$ of restriction semigroups such that $\psi_2\gamma = \psi_1$.
- If T and S are inverse, we obtain the category of idempotent pure extensions of S considered by Lawson (1996).
- Let $(\alpha, p, X) \in \widehat{\mathcal{A}}(S)$. The assignment $(\alpha, p, X) \rightsquigarrow X \rtimes_\alpha^p S$ gives rise to a functor $\widehat{U}\colon \widehat{\mathcal{A}}(S) \to \mathcal{P}(S)$.
- Let $\psi\colon T \to S \in \mathcal{P}(S)$. The assignment $\psi \rightsquigarrow (\widehat{\psi}, \psi|_{P(T)}, S)$ gives rise to a functor $\widehat{G}\colon \mathcal{P}(S) \to \widehat{\mathcal{A}}(S)$.
- Theorem (M. Dokuchaev, M. Khrypchenko, GK, 2019). The functors $\widehat{U}$ and $\widehat{G}$ establish an equivalence between the categories $\widehat{\mathcal{A}}(S)$ and $\mathcal{P}(S)$.
- Corollary. Let S be a monoid or, in particular, a group. Then the categories $\mathcal{P}(S)$ and $\mathcal{A}(S)$ are equivalent.