# On the function sum of element orders in a finite group

Mercede MAJ

UNIVERSITÀ DEGLI STUDI DI SALERNO

SandGAL 2019
Semigroups and Groups, Automata, Logics

*Politecnico di Milano - Cremona Campus*

June 10-13, 2019

# Dedication

*Dedicated to* Sandra Cherubini

Marcel Herzog
(Tel Aviv)

Patrizia Longobardi
(Salerno)

**Marcel Herzog, Patrizia Longobardi, M. M.**

# Papers

Marcel Herzog
(Tel Aviv)

Mercede Maj
(Salerno)

**Marcel Herzog, P. L., Mercede Maj**

# Papers


- M. Herzog, P. Longobardi, M. Maj, An exact upper bound for sums of element orders in non-cyclic finite groups, *J. Pure Appl. Algebra*, **222** n. 7 (2018), 1628-1642.
- M. Herzog, P. Longobardi, M. Maj, Sums of element orders in groups of order $2m$ with $m$ odd, *Comm. Algebra*, to appear. doi.org/10.1080/00927872.2018.1527924,
- M. Herzog, P. Longobardi, M. Maj, Two criteria for solvability of finite groups, *J. Algebra*, **511** (2018), 215-226.
- M. Herzog, P. Longobardi, M. Maj, The second maximal groups with respect to the sum of elements orders, submitted, arxiv.org/abs/1901.09662.

# Papers

- M. Herzog, P. Longobardi, M. Maj, Sums of element orders in groups of odd order, in preparation.

- M. Herzog, P. Longobardi, M. Maj, Properties of finite and periodic groups determined by their elements orders (a survey), *Group Theory and Computation, Indian Statistical Institute Series*, Editors N.S. Narasimha Sastry, Manoj Kumar Yadav, (2018), 59-90.

# Papers

- M. Herzog, P. Longobardi, M. Maj, Sums of element orders in groups of odd order, in preparation.
- M. Herzog, P. Longobardi, M. Maj, Properties of finite and periodic groups determined by their elements orders (a survey), *Group Theory and Computation, Indian Statistical Institute Series*, Editors N.S. Narasimha Sastry, Manoj Kumar Yadav, (2018), 59-90.

# Basic Problem

Let $G$ be a periodic group.

## Main Problem

*Obtain information about the structure of $G$*

*by looking at the orders of its elements.*

# The function $\psi$

Let $G$ be a finite group.

**Definition**

$$\psi(G) := \sum_{x \in G} o(x).$$

**Problem**

*What can be said about the structure of $G$ by looking at the value $\psi(G)$?*

# The function $\psi$

Let $G$ be a finite group.

**Definition**

$$\psi(G) := \sum_{x \in G} o(x).$$

**Problem**

*What can be said about the structure of $G$*

*by looking at the value $\psi(G)$?*

# The function $\psi$

Let $G$ be a finite group.

## Definition

$$\psi(G) := \sum_{x \in G} o(x).$$

## Problem

*What can be said about the structure of $G$*

*by looking at the value $\psi(G)$?*

**Examples**

$$\psi(\mathcal{S}_3) = 13.$$

For, $\psi(\mathcal{S}_3) = 1 \cdot 1 + 3 \cdot 2 + 2 \cdot 3$.

$$\psi(\mathcal{C}_6) = 21.$$

For, $\psi(\mathcal{C}_6) = 1 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + 2 \cdot 6$.

$$\psi(\mathcal{C}_5) = 21.$$

For, $\psi(\mathcal{C}_5) = 1 \cdot 1 + 4 \cdot 5$.

where $\mathcal{C}_n$ is the cyclic group of order $n$ and $\mathcal{S}_3$ is the symmetric group of degree 3.

# The function $\psi$ - some examples

## Examples

$$\psi(\mathcal{S}_3) = 13.$$

For, $\psi(\mathcal{S}_3) = 1 \cdot 1 + 3 \cdot 2 + 2 \cdot 3.$

$$\psi(\mathcal{C}_6) = 21.$$

For, $\psi(\mathcal{C}_6) = 1 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + 2 \cdot 6.$

$$\psi(\mathcal{C}_5) = 21.$$

For, $\psi(\mathcal{C}_5) = 1 \cdot 1 + 4 \cdot 5.$

where $\mathcal{C}_n$ is the cyclic group of order $n$ and $\mathcal{S}_3$ is the symmetric group of degree 3.

# The function $\psi$ - some examples

## Examples

$$\psi(\mathcal{S}_3) = 13.$$

For, $\psi(\mathcal{S}_3) = 1 \cdot 1 + 3 \cdot 2 + 2 \cdot 3.$

$$\psi(\mathcal{C}_6) = 21.$$

For, $\psi(\mathcal{C}_6) = 1 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + 2 \cdot 6.$

$$\psi(\mathcal{C}_5) = 21.$$

For, $\psi(\mathcal{C}_5) = 1 \cdot 1 + 4 \cdot 5.$

where $\mathcal{C}_n$ is the cyclic group of order $n$ and $\mathcal{S}_3$ is the symmetric group of degree 3.

# The function $\psi$ - some remarks

**Remark**

$\psi(G) = \psi(G_1)$ does not imply $G \simeq G_1$.

**Example**

Let $A = C_8 \times C_2$,
$B = C_2 \ltimes C_8$, where $C_2 = \langle a \rangle$, $C_8 = \langle b \rangle$, $b^a = b^5$.
Then
$\psi(A) = \psi(B) = 87$.

**Remark**

$|G| = |G_1|$ and $\psi(G) = \psi(G_1)$ do not imply $G \simeq G_1$.

# The function $\psi$ - some remarks

**Remark**

$$\psi(G) = \psi(G_1) \text{ does not imply } G \simeq G_1.$$

**Example**

Let $A = C_8 \times C_2$,
$B = C_2 \ltimes C_8$, where $C_2 = \langle a \rangle$, $C_8 = \langle b \rangle$, $b^a = b^5$.
Then
$\psi(A) = \psi(B) = 87$.

**Remark**

$$|G| = |G_1| \text{ and } \psi(G) = \psi(G_1) \text{ do not imply } G \simeq G_1.$$

# The function $\psi$ - a problem

**Remark**

$\psi(G) = \psi(\mathcal{S}_3)$ implies $G \simeq \mathcal{S}_3$.

**Problem**

*Find information about the structure of a finite group $G$ from some inequalities on $\psi(G)$.*

## The function $\psi$ - a property

**Proposition**

If $G = G_1 \times G_2$, where $|G_1|$ and $|G_2|$ are coprime, then

$$\psi(G) = \psi(G_1)\psi(G_2).$$

## The function $\psi$ - a property

**Proposition**

If $G = G_1 \times G_2$, where $|G_1|$ and $|G_2|$ are coprime, then

$$\psi(G) = \psi(G_1)\psi(G_2).$$

# Sum of the orders of the elements in a cyclic group

**Remark**

$$\psi(C_n) = \sum_{d|n} d\varphi(d),$$

where $\varphi$ is the Euler's function.

**Proposition**

Let $p$ be a prime, $\alpha \geq 0$. Then:

$\psi(C_{p^\alpha}) = \frac{p^{2\alpha+1}+1}{p+1}$.

# Sum of the orders of the elements in a cyclic group

**Remark**

$$\psi(C_n) = \sum_{d|n} d\varphi(d),$$

where $\varphi$ is the Euler's function.

**Proposition**

Let $p$ be a prime, $\alpha \geq 0$. Then:
$\psi(C_{p^\alpha}) = \frac{p^{2\alpha+1}+1}{p+1}$.

# Sum of the orders of the elements in a cyclic group

**Proposition**

Let $p$ be a prime, $\alpha \geq 0$. Then: $\psi(\mathcal{C}_{p^\alpha}) = \frac{p^{2\alpha+1}+1}{p+1}$.

*Proof.* $\psi(\mathcal{C}_{p^\alpha}) = 1 + p\varphi(p) + p^2\varphi(p^2) + \cdots + p^\alpha(\varphi(p^\alpha)) =$

$1 + p(p-1) + p^2(p^2 - p) + \cdots + p^\alpha(p^\alpha - p^{\alpha-1}) =$

$= 1 + p^2 - p + p^4 - p^3 + \cdots + p^{2\alpha} - p^{2\alpha-1} = \frac{p^{2\alpha+1}+1}{p+1}$, as required.//

**Corollary**

Let $n > 1$. Write $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, $p_i's$ different primes, $\alpha_i$'s $> 0$. Then

$$\psi(\mathcal{C}_n) = \prod_{i \in \{1,\cdots,s\}} \frac{p_i^{2\alpha_i+1}+1}{p_i+1}.$$

# Sum of the orders of the elements in a cyclic group

**Proposition**

Let $p$ be a prime, $\alpha \geq 0$. Then: $\psi(C_{p^\alpha}) = \frac{p^{2\alpha+1}+1}{p+1}$.

*Proof.* $\psi(C_{p^\alpha}) = 1 + p\varphi(p) + p^2\varphi(p^2) + \cdots + p^\alpha(\varphi(p^\alpha)) =$

$1 + p(p-1) + p^2(p^2-p) + \cdots + p^\alpha(p^\alpha - p^{\alpha-1}) =$

$= 1 + p^2 - p + p^4 - p^3 + \cdots + p^{2\alpha} - p^{2\alpha-1} = \frac{p^{2\alpha+1}+1}{p+1}$, as required.//

**Corollary**

Let $n > 1$. Write $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, $p_i's$ different primes, $\alpha_i$'s $> 0$. Then

$$\psi(C_n) = \prod_{i \in \{1,\cdots,s\}} \frac{p_i^{2\alpha_i+1}+1}{p_i+1}.$$

# The function $\psi$ - a first result

**Theorem [H. Amiri, S.M. Jafarian Amiri, M. Isaacs]**

Let $G$ be a finite group, $|G| = n$. Then

$$\psi(G) \leq \psi(C_n).$$

Moreover

$$\psi(G) = \psi(C_n) \text{ if and only if } G \simeq C_n.$$

H. Amiri, S.M. Jafarian Amiri, I.M. Isaacs, Sums of element orders in finite groups, *Comm. Algebra* **37** (2009), 2978-2980.

# The function $\psi$ - a first result

**Theorem** [**H. Amiri, S.M. Jafarian Amiri, M. Isaacs**]

Let $G$ be a finite group, $|G| = n$. Then

$$\psi(G) \leq \psi(C_n).$$

Moreover

$\psi(G) = \psi(C_n)$ if and only if $G \simeq C_n$.

H. Amiri, S.M. Jafarian Amiri, I.M. Isaacs, Sums of element orders in finite groups, *Comm. Algebra* **37** (2009), 2978-2980.

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a non-cyclic group of order $n$. Then

$$\psi(G) \leq \frac{7}{11}\psi(C_n).$$

Moreover

this bound is best possible.

M. Herzog, P. Longobardi, M. Maj, An exact upper bound for sums of element orders in non-cyclic finite groups, *J. Pure Appl. Algebra*, **222** n. 7 (2018), 1628-1642.

# The function $\psi$ - some recent results

**Theorem** [M. Herzog, P. L., M. M.]

Let $G$ be a non-cyclic group of order $n$. Then

$$\psi(G) \leq \frac{7}{11}\psi(C_n).$$

Moreover

this bound is best possible.

M. Herzog, P. Longobardi, M. Maj, An exact upper bound for sums of element orders in non-cyclic finite groups, *J. Pure Appl. Algebra*, **222** n. 7 (2018), 1628-1642.

### Remark

The upper bound $\frac{7}{11}$ is best possible.

For example,

$\psi(C_2 \times C_2) = 7$ and $\psi(C_4) = 11$. Therefore

$$\psi(C_2 \times C_2) = \frac{7}{11}\psi(C_4).$$

Moreover,

it is easy to see that if $n = 4k$ for some odd integer $k$, then

the group $G = C_{2k} \times C_2$ satisfies the above equality.

**Remark**

The upper bound $\frac{7}{11}$ is best possible.

For example,

$\psi(\mathcal{C}_2 \times \mathcal{C}_2) = 7$ and $\psi(\mathcal{C}_4) = 11$. Therefore

$$\psi(\mathcal{C}_2 \times \mathcal{C}_2) = \frac{7}{11}\psi(\mathcal{C}_4).$$

Moreover,

it is easy to see that if $n = 4k$ for some odd integer $k$, then

the group $G = \mathcal{C}_{2k} \times \mathcal{C}_2$ satisfies the above equality.

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a non-cyclic group of order $n$. Then

$$\psi(G) = \frac{7}{11}\psi(C_n)$$

if and only if

$n = 4k$ with $(k, 2) = 1$ and $G = (C_2 \times C_2) \times C_k$.

M. Herzog, P. Longobardi, M. Maj, The second maximal groups with respect to the sum of element orders, submitted, arxiv.org/abs/1901.09662.

# The function $\psi$ - a new result

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a non-cyclic group of order $n$. Then

$$\psi(G) \leq \frac{7}{11}\psi(C_n).$$

Moreover $\psi(G) = \frac{7}{11}\psi(C_n)$ if and only if

$n = 4k$ with $(k, 2) = 1$ and $G = (C_2 \times C_2) \times C_k$.

M. Herzog, P. Longobardi, M. Maj, The second maximal groups with respect to the sum of element orders, submitted, arxiv.org/abs/1901.09662.

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a non-cyclic group of order $n$ and let $q$ be the smallest prime divisor of $n$. Then

$$\psi(G) < \frac{1}{q-1}\psi(C_n).$$

**Corollary**

Let $G$ be a non-cyclic finite group of odd order $n$. Then

$$\psi(G) < \frac{1}{2}\psi(C_n).$$

# The function $\psi$ - some recent results

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a non-cyclic group of order $n$ and let $q$ be the smallest prime divisor of $n$. Then

$$\psi(G) < \frac{1}{q-1}\psi(C_n).$$

**Corollary**

Let $G$ be a non-cyclic finite group of odd order $n$. Then

$$\psi(G) < \frac{1}{2}\psi(C_n).$$

# The function $\psi$ - a new result

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a non-cyclic group of order $n$ and let $q$ be the smallest prime divisor of $n$. Then

$$\psi(G) \leq \frac{((q^2-1)q+1)(q+1)}{q^5+1}\psi(C_n)$$

and the equality holds if and only if

$$n = q^2k \text{ with } (k, q!) = 1 \text{ and } G = (C_q \times C_q) \times C_k.$$

M. Herzog, P. Longobardi, M. Maj, The second maximal groups with respect to the sum of element orders, submitted, arxiv.org/abs/1901.09662.

## The function $\psi$ - a new result

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a non-cyclic group of order $n$ and let $q$ be the smallest prime divisor of $n$. Then

$$\psi(G) \leq \frac{((q^2-1)q+1)(q+1)}{q^5+1}\psi(C_n)$$

and the equality holds if and only if

$$n = q^2k \text{ with } (k, q!) = 1 \text{ and } G = (C_q \times C_q) \times C_k.$$

M. Herzog, P. Longobardi, M. Maj, The second maximal groups with respect to the sum of element orders, submitted, arxiv.org/abs/1901.09662.

# The function $\psi$ - a new result

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a non-cyclic group of order $n$ and let $q$ be the smallest prime divisor of $n$. Then

$$\psi(G) \leq \frac{((q^2-1)q+1)(q+1)}{q^5+1}\psi(C_n)$$

and the equality holds if and only if

$$n = q^2 k \text{ with } (k, q!) = 1 \text{ and } G = (C_q \times C_q) \times C_k.$$

Notice that for $q = 2$ we have:

$$\frac{((2^2-1)2+1)(2+1)}{2^5+1} = \frac{(3 \cdot 2+1)3}{33} = \frac{7}{11}.$$

## The function $\psi$ - a new result - some ingredients

**Theorem [H. Amiri, S.M. Jafarian Amiri, I.M. Isaacs]**

Let $G$ be a finite group, $p$ a prime, $P$ a normal cyclic Sylow $p$-subgroup of $G$. Then

$$\psi(G) \leq \psi(G/P)\psi(P),$$

with equality if and only if $P$ is central in $G$.

**Theorem [M. Herzog, P. L., M. M., 2018]**

Let $n$ be a positive integer and let $q$ be the smallest prime divisor of $n$, $p$ the largest prime divisor of $n$. Then

$$\psi(C_n) \geq \frac{q}{p+1}n^2.$$

## The function $\psi$ - a new result - some ingredients

**Theorem [H. Amiri, S.M. Jafarian Amiri, I.M. Isaacs]**

Let $G$ be a finite group, $p$ a prime, $P$ a normal cyclic Sylow $p$-subgroup of $G$. Then

$$\psi(G) \leq \psi(G/P)\psi(P),$$

with equality if and only if $P$ is central in $G$.

**Theorem [M. Herzog, P. L., M. M., 2018]**

Let $n$ be a positive integer and let $q$ be the smallest prime divisor of $n$, $p$ the largest prime divisor of $n$. Then

$$\psi(C_n) \geq \frac{q}{p+1}n^2.$$

# The function $\psi$ - a new result - some ingredients

**Theorem** [**H. Amiri, S.M. Jafarian Amiri, I.M. Isaacs**]

Let $G$ be a finite group, $p$ a prime,
$P$ a normal cyclic Sylow $p$-subgroup of $G$. Then

$$\psi(G) \leq \psi(G/P)\psi(P),$$

with equality if and only if $P$ is central in $G$.

**Theorem** [**M. Herzog, P. L., M. M., 2018**]

Let $n$ be a positive integer and let $q$ be the smallest prime divisor of $n$, $p$ the largest prime divisor of $n$.
Then

$$\psi(C_n) \geq \frac{q}{p+1} n^2.$$

# The function $\psi$ - a new result - some ingredients

**Theorem** [M. Herzog, P. L., M. M., 2018]

Let $G = P \rtimes F$, where $P$ is a cyclic Sylow $p$-subgroup, $|F| > 1$ and $Z = C_F(P)$. Then

$$\psi(G) < \psi(P)\psi(F)\left(\frac{\psi(Z)}{\psi(F)} + \frac{|P|}{\psi(P)}\right).$$

**Theorem** [M. Herzog, P. L., M. Maj, 2019]

Let $G = P \rtimes F$, where $P$ is a cyclic Sylow $p$-subgroup of $G$ and $F$ is a cyclic $p$-complement. If $\psi(G)$ has the second largest value for groups of order $|G|$, then

$C_F(P)$ is a maximal subgroup of $F$.

# The function $\psi$ - a new result - some ingredients

**Theorem [M. Herzog, P. L., M. M., 2018]**

Let $G = P \rtimes F$, where $P$ is a cyclic Sylow $p$-subgroup, $|F| > 1$ and $Z = C_F(P)$. Then

$$\psi(G) < \psi(P)\psi(F)\left(\frac{\psi(Z)}{\psi(F)} + \frac{|P|}{\psi(P)}\right).$$

**Theorem [M. Herzog, P. L., M. Maj, 2019]**

Let $G = P \rtimes F$, where $P$ is a cyclic Sylow $p$-subgroup of $G$ and $F$ is a cyclic $p$-complement. If $\psi(G)$ has the second largest value for groups of order $|G|$, then

$C_F(P)$ is a maximal subgroup of $F$.

## The function $\psi$ - a new result

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a non-cyclic group of order $n$. Then

$$\psi(G) \leq \frac{7}{11}\psi(C_n).$$

Moreover $\psi(G) = \frac{7}{11}\psi(C_n)$ if and only if

$n = 4k$ with $(k, 2) = 1$ and $G = (C_2 \times C_2) \times C_k$.

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a non-cyclic group of order $n = 2m$, with $m$ an odd integer. Then

$$\psi(G) \leq \frac{13}{21}\psi(C_n).$$

Moreover

$$\psi(G) = \frac{13}{21}\psi(C_n) \quad \text{if and only if} \quad G = S_3 \times C_{n/6},$$

where $n = 6m_1$ with $(m_1, 6) = 1$ and $S_3$ is the symmetric group on three letters.

M. Herzog, P. Longobardi, M. Maj, Sums of element orders in groups of order $2m$ with $m$ odd, submitted.

# The function $\psi$ - some recent results - the even case

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a non-cyclic group of order $n = 2m$, with $m$ an odd integer. Then

$$\psi(G) \leq \frac{13}{21}\psi(\mathcal{C}_n).$$

Moreover

$$\psi(G) = \frac{13}{21}\psi(\mathcal{C}_n) \quad \text{if and only if} \quad G = \mathcal{S}_3 \times \mathcal{C}_{n/6},$$

where $n = 6m_1$ with $(m_1, 6) = 1$ and $\mathcal{S}_3$ is the symmetric group on three letters.

M. Herzog, P. Longobardi, M. Maj, Sums of element orders in groups of order $2m$ with $m$ odd, submitted.

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a non-cyclic group of even order $n = 2^\alpha m$, with $m$ an odd integer. Then

(i) if $\alpha = 1$, then $\frac{\psi(G)}{\psi(C_n)} \leq \frac{13}{21} = \frac{\psi(S_3)}{\psi(C_6)}$,

(ii) if $\alpha = 2$, then $\frac{\psi(G)}{\psi(C_n)} \leq \frac{7}{11} = \frac{\psi(C_2 \times C_2)}{\psi(C_4)}$,

In particular, these upper bounds are best possible.

# The function $\psi$ - some recent results - the even case

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a non-cyclic group of even order $n = 2^\alpha m$, with $m$ an odd integer. Then

(i) if $\alpha = 1$, then $\frac{\psi(G)}{\psi(C_n)} \leq \frac{13}{21} = \frac{\psi(S_3)}{\psi(C_6)}$,

(ii) if $\alpha = 2$, then $\frac{\psi(G)}{\psi(C_n)} \leq \frac{7}{11} = \frac{\psi(C_2 \times C_2)}{\psi(C_4)}$,

In particular, these upper bounds are best possible.

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a non-cyclic group of even order $n = 2^{\alpha} m$, with $m$ an odd integer. Then

(i) if $\alpha = 1$, then $\frac{\psi(G)}{\psi(C_n)} \leq \frac{13}{21} = \frac{\psi(S_3)}{\psi(C_6)}$,

(ii) if $\alpha = 2$, then $\frac{\psi(G)}{\psi(C_n)} \leq \frac{7}{11} = \frac{\psi(C_2 \times C_2)}{\psi(C_4)}$,

In particular, these upper bounds are best possible.

# The function $\psi$ - some recent results - the even case

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a non-cyclic group of even order $n = 2^\alpha m$, with $m$ an odd integer. Then

(i) if $\alpha = 1$, then $\frac{\psi(G)}{\psi(C_n)} \leq \frac{13}{21} = \frac{\psi(S_3)}{\psi(C_6)}$,

(ii) if $\alpha = 2$, then $\frac{\psi(G)}{\psi(C_n)} \leq \frac{7}{11} = \frac{\psi(C_2 \times C_2)}{\psi(C_4)}$,

In particular, these upper bounds are best possible.

# The function $\psi$ - some recent results - the even case

**Theorem [S.M. Jafarian Amiri, M. Amiri, 2014; R. Shen, G. Chen, C. Wu, 2015]**

Let $G$ be a non-cyclic group of even order $n = 2^{\alpha}m$, with $m$ an odd integer. Then

(iii) if $\alpha = 3$, then $\frac{\psi(G)}{\psi(C_n)} \leq \frac{27}{43} = \frac{\psi(Q_8)}{\psi(C_8)}$,

(iv) if $\alpha \geq 4$, then $\frac{\psi(G)}{\psi(C_n)} \leq \frac{2^{2\alpha+3}+7}{7(2^{\alpha+1})+1} = \frac{\psi(G_1)}{\psi(C_{2^{\alpha}\cdot 3})}$,

where $Q_8$ is the quaternion group of order 8, and $G_1 = \langle a, b \mid a^{2^{\alpha}} = b^3 = 1, b^a = b^{-1} \rangle$.

In particular, these upper bounds are best possible.

**Theorem [S.M. Jafarian Amiri, M. Amiri, 2014; R. Shen, G. Chen, C. Wu, 2015]**

Let $G$ be a non-cyclic group of even order $n = 2^\alpha m$, with $m$ an odd integer. Then

(iii) if $\alpha = 3$, then $\frac{\psi(G)}{\psi(C_n)} \leq \frac{27}{43} = \frac{\psi(Q_8)}{\psi(C_8)}$,

(iv) if $\alpha \geq 4$, then $\frac{\psi(G)}{\psi(C_n)} \leq \frac{2^{2\alpha+3}+7}{7(2^{\alpha+1}))+1} = \frac{\psi(G_1)}{\psi(C_{2^\alpha \cdot 3})}$,

where $Q_8$ is the quaternion group of order 8, and $G_1 = \langle a, b \mid a^{2^\alpha} = b^3 = 1, b^a = b^{-1} \rangle$.

In particular, these upper bounds are best possible.

# The function $\psi$ - some recent results - the even case

**Theorem [S.M. Jafarian Amiri, M. Amiri, 2014; R. Shen, G. Chen, C. Wu, 2015]**

Let $G$ be a non-cyclic group of even order $n = 2^{\alpha} m$, with $m$ an odd integer. Then

(iii) if $\alpha = 3$, then $\frac{\psi(G)}{\psi(C_n)} \leq \frac{27}{43} = \frac{\psi(Q_8)}{\psi(C_8)}$,

(iv) if $\alpha \geq 4$, then $\frac{\psi(G)}{\psi(C_n)} \leq \frac{2^{2\alpha+3}+7}{7(2^{\alpha+1})+1} = \frac{\psi(G_1)}{\psi(C_{2^{\alpha} \cdot 3})}$,

where $Q_8$ is the quaternion group of order 8, and $G_1 = \langle a, b \mid a^{2^{\alpha}} = b^3 = 1, b^a = b^{-1} \rangle$.

In particular, these upper bounds are best possible.

# The function $\psi$ - some recent results

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a non-cyclic group of order $n$ and let $q$ be the smallest prime divisor of $n$. Then

$$\psi(G) < \frac{1}{q-1}\psi(C_n).$$

**Remark**

It is not possible to substitute $q-1$ by $q$. For:

$$\psi(S_3) = 13 \geq \frac{1}{2}\psi(C_6) = \frac{21}{2}.$$

# The function $\psi$ - some recent results

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a non-cyclic group of order $n$ and let $q$ be the smallest prime divisor of $n$. Then

$$\psi(G) < \frac{1}{q-1}\psi(C_n).$$

**Remark**

It is not possible to substitute $q-1$ by $q$. For:

$$\psi(S_3) = 13 \geq \frac{1}{2}\psi(C_6) = \frac{21}{2}.$$

**Theorem [M. Herzog, P. L., M. M., J. Pure Appl. Algebra 2018]**

Let $G$ be a finite group, $|G| = n$ and
let $q$ be the smallest prime divisor of $n$.

If $\psi(G) \geq \frac{1}{q}\psi(C_n)$, then

$G$ is soluble and $G'' \leq Z(G)$.

**Remark**

Hence if $G$ is a finite non soluble group of order $n$ and $q$ is the smallest prime divisor of $n$, then

$$\psi(G) < \frac{1}{q}\psi(C_n).$$

**Theorem** [M. Herzog, P. L., M. M., J. Pure Appl. Algebra 2018]

Let $G$ be a finite group, $|G| = n$ and
let $q$ be the smallest prime divisor of $n$.

If $\psi(G) \geq \frac{1}{q}\psi(C_n)$, then

$G$ is soluble and $G'' \leq Z(G)$.

**Remark**

Hence if $G$ is a finite non soluble group of order $n$ and
$q$ is the smallest prime divisor of $n$, then

$\psi(G) < \frac{1}{q}\psi(C_n)$.

**Theorem** [**M. Herzog, P. L., M. M., J. Pure Appl. Algebra 2018**]

Let $G$ be a finite group, $|G| = n$ and
let $q$ be the smallest prime divisor of $n$.

If $\psi(G) \geq \frac{1}{q}\psi(C_n)$, then

$G$ is soluble and $G'' \leq Z(G)$.

**Remark**

Hence if $G$ is a finite non soluble group of order $n$ and $q$ is the smallest prime divisor of $n$, then

$$\psi(G) < \frac{1}{q}\psi(C_n).$$

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a finite group of order $n$ and suppose that

$$\psi(G) \geq \frac{1}{6.68}\psi(C_n).$$

Then

$G$ is a solvable group.

**Remark**

Hence if $G$ is a finite non soluble group of order $n$, then

$$\psi(G) < \frac{1}{6.68}\psi(C_n).$$

**Theorem** [**M. Herzog, P. L., M. M.**]

Let $G$ be a finite group of order $n$ and suppose that

$$\psi(G) \geq \frac{1}{6.68}\psi(C_n).$$

Then

$G$ is a solvable group.

**Remark**

Hence if $G$ is a finite non soluble group of order $n$, then

$$\psi(G) < \frac{1}{6.68}\psi(C_n).$$

# The function $\psi$ - a conjecture

**Remark**

Notice that $\psi(\mathcal{A}_5) = 211$ and $\psi(C_{60}) = 1617$. Therefore

$$\psi(\mathcal{A}_5) = \frac{211}{1617}\psi(C_{60}) > \frac{1}{7.67}\psi(C_{60}).$$

So our lower bound $\frac{1}{6.68}$ in the previous Theorem is not very far from the best possible one.

**Conjecture**

If $G$ is a group of order $n$ and

$$\psi(G) > \frac{211}{1617}\psi(C_n),$$

then $G$ is solvable.

If true, this lower bound is certainly best possible.

**Remark**

Notice that $\psi(\mathcal{A}_5) = 211$ and $\psi(C_{60}) = 1617$. Therefore

$$\psi(\mathcal{A}_5) = \frac{211}{1617}\psi(C_{60}) > \frac{1}{7.67}\psi(C_{60}).$$

So our lower bound $\frac{1}{6.68}$ in the previous Theorem is not very far from the best possible one.

**Conjecture**

If $G$ is a group of order $n$ and

$$\psi(G) > \frac{211}{1617}\psi(C_n),$$

then $G$ is solvable.

If true, this lower bound is certainly best possible.

# The function $\psi$ - validity of the conjecture

**Theorem [M. Baniasad Asad, B. Khosravi]**

If $G$ is a group of order $n$ and

$$\psi(G) > \frac{211}{1617}\psi(C_n),$$

then $G$ is solvable.

Moreover, if $G = A_5 \times C_m$, where $(30, m) = 1$,

then $\psi(G) = \frac{211}{1617}\psi(C_n)$ .

M. Baniasad Asad, B. Khosravi, A Criterion for Solvability of a Finite Group by the Sum of Element Orders, *J. Algebra* **516** (2018), 115-124.

**Theorem [M. Baniasad Asad, B. Khosravi]**

If $G$ is a group of order $n$ and

$$\psi(G) > \frac{211}{1617}\psi(C_n),$$

then $G$ is solvable.

Moreover, if $G = \mathcal{A}_5 \times C_m$, where $(30, m) = 1$,

then $\psi(G) = \frac{211}{1617}\psi(C_n)$ .

M. Baniasad Asad, B. Khosravi, A Criterion for Solvability of a Finite Group by the Sum of Element Orders, *J. Algebra* **516** (2018), 115-124.

**Theorem [A. Bahri, B. Khosravi, Z. Akhlaghi]**

If $G$ is a non-solvable group of order $n$ and

$$\psi(G) = \frac{211}{1617}\psi(C_n),$$

then $G = A_5 \times C_m$, where is $(30, m) = 1$.

A. Bahri, B. Khosravi, Z. Akhlaghi, A result on the sum of element orders of a finite group, arXiv:1904.00425, March 2019.

# The function $\psi$ - some recent results - the even case

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a non-cyclic group of order $n = 2m$, with $m$ an odd integer. Then

$$\psi(G) \leq \frac{13}{21}\psi(C_n).$$

**Theorem [M. Tărnăuceanu]**

Let $G$ be a group of order $n$ with $\psi(G) > \frac{13}{21}\psi(C_n)$.

Then $G$ is nilpotent.

M. Tărnăuceanu, A criterion for nilpotency of a finite group by the sum of element orders, submitted, arXiv:1903.09744.

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a non-cyclic group of order $n = 2m$, with $m$ an odd integer. Then

$$\psi(G) \leq \frac{13}{21}\psi(C_n).$$

**Theorem [M. Tărnăuceanu]**

Let $G$ be a group of order $n$ with $\psi(G) > \frac{13}{21}\psi(C_n)$.

Then $G$ is nilpotent.

M. Tărnăuceanu, A criterion for nilpotency of a finite group by the sum of element orders, submitted, arXiv:1903.09744.

# The function $\psi$ - some recent results - the even case

**Theorem [M. Herzog, P. L., M. M.]**

Let $G$ be a non-cyclic group of order $n = 2m$, with $m$ an odd integer. Then

$$\psi(G) \leq \frac{13}{21}\psi(\mathcal{C}_n).$$

**Theorem [M. Tărnăuceanu]**

Let $G$ be a group of order $n$ with $\psi(G) > \frac{13}{21}\psi(\mathcal{C}_n)$.

Then $G$ is nilpotent.

M. Tărnăuceanu, A criterion for nilpotency of a finite group by the sum of element orders, submitted, arXiv:1903.09744.

M. Tărnăuceanu, D.G. Fodor considered the class of all finite abelian groups $G$ and gave an explicit formula for $\psi(G)$.

M. Tărnăuceanu, D.G. Fodor, On the sum of element orders of finite abelian groups, *Sci. An. Univ. "A1.I. Cuza" Iasi, Ser. Math.*, **XL** (2014), 1-7.

S.M. Jafarian Amiri and M. Amiri considered the class of all finite $p$-groups and all groups with square-free order.

S.M. Jafarian Amiri, M. Amiri, Characterization of $p$-groups by sum of the element orders, *Publ. Math. Debrecen* **86** no. 1-2 (2015), 31-37.

S.M. Jafarian Amiri, M. Amiri, Sum of the Element Orders in Groups with the Square-Free Order, *Bull. Malays. Math. Sci. Soc.* **40** (2017), 1025-1034.

. . .

# Sum of the orders of the elements

## Definition

Let $n$ be a positive integer. Put

$$\mathcal{T}_n := \{\psi(H) \mid |H| = n\}$$

Recall that

$\psi(C_n)$ is the **maximum** of $\mathcal{T}_n$.

## Problem

*What is the structure of $G$ if $\psi(G)$ is the* **minimum** *of $\mathcal{T}_n$?*

## Definition

Let $n$ be a positive integer. Put

$$\mathcal{T}_n := \{\psi(H) \mid |H| = n\}$$

Recall that

$\psi(C_n)$ is the **maximum** of $\mathcal{T}_n$.

## Problem

*What is the structure of G if $\psi(G)$ is the* **minimum** *of $\mathcal{T}_n$?*

# Sum of the orders of the elements

## Definition

Let $n$ be a positive integer. Put

$$\mathcal{T}_n := \{\psi(H) \mid |H| = n\}$$

Recall that

$\psi(C_n)$ is the **maximum** of $\mathcal{T}_n$.

## Problem

*What is the structure of $G$ if $\psi(G)$ is the* **minimum** *of $\mathcal{T}_n$?*

## Remarks

If $n = p^\alpha$ for some prime $p$ and some $\alpha > 0$ and $|G| = p^\alpha$,
then obviously
$\psi(G)$ is minimum if and only if $expG = p$.

If $p = 2$ and $\psi(G)$ is minimum ,
then $G$ is the elementary abelian group of order $2^\alpha$.

But there are non-isomorphic groups $G$ and $G_1$ of order $p^\alpha > p^2$ ( $p > 2$)
with $\psi(G) = \psi(G_1)$ minimum.

For instance, the two groups of exponent 3 and order $3^3$.

# Sum of the orders of the elements - minimum

**Remarks**

If $n = p^\alpha$ for some prime $p$ and some $\alpha > 0$ and $|G| = p^\alpha$, then obviously $\psi(G)$ is minimum if and only if $expG = p$.

If $p = 2$ and $\psi(G)$ is minimum , then $G$ is the elementary abelian group of order $2^\alpha$.

But there are non-isomorphic groups $G$ and $G_1$ of order $p^\alpha > p^2$ ( $p > 2$) with $\psi(G) = \psi(G_1)$ minimum.

For instance, the two groups of exponent 3 and order $3^3$.

## Remarks

If $n = p^\alpha$ for some prime $p$ and some $\alpha > 0$ and $|G| = p^\alpha$, then obviously $\psi(G)$ is minimum if and only if $expG = p$.

If $p = 2$ and $\psi(G)$ is minimum , then $G$ is the elementary abelian group of order $2^\alpha$.

But there are non-isomorphic groups $G$ and $G_1$ of order $p^\alpha > p^2$ ( $p > 2$) with $\psi(G) = \psi(G_1)$ minimum.

For instance, the two groups of exponent 3 and order $3^3$.

# Sum of the orders of the elements - minimum

**Problem**

*What happens in the general case?*

*What happens for non-abelian simple groups?*

**Theorem [H. Amiri, S.M. Jafarian Amiri, 2011]**

Let $G$ be a finite nilpotent group of order $n$

and assume that there are non-nilpotent groups of order $n$.

Then there exists a non-nilpotent group $K$ with $|K| = |G|$ such that

$$\psi(K) < \psi(G).$$

# Sum of the orders of the elements - minimum

**Problem**

*What happens in the general case?*

*What happens for non-abelian simple groups?*

**Theorem [H. Amiri, S.M. Jafarian Amiri, 2011]**

Let $G$ be a finite nilpotent group of order $n$

and assume that there are non-nilpotent groups of order $n$.

Then there exists a non-nilpotent group $K$ with $|K| = |G|$ such that

$$\psi(K) < \psi(G).$$

# Sum of the orders of the elements - minimum

**Question**

If $S$ is a simple group of order $n$,
is $\psi(S)$ the minimum of $\mathcal{T}_n$?

**NO!**

There are non-isomorphic simple groups $S$ and $S_1$ such that $|S| = |S_1|$ and $\psi(S) \neq \psi(S_1)$.

For instance, the groups $\mathcal{A}_8$ and $\mathcal{PSL}(3,4)$ are such that $|\mathcal{A}_8| = 20160 = |\mathcal{PSL}(3,4)|$ and $\psi(\mathcal{A}_8) = 137047 > 103111 = \psi(\mathcal{PSL}(3,4))$.

**Question**

If $S$ is a simple group of order $n$,
is $\psi(S)$ the minimum of $\mathcal{T}_n$?

**NO!**

There are non-isomorphic simple groups $S$ and $S_1$ such that $|S| = |S_1|$ and $\psi(S) \neq \psi(S_1)$.

For instance, the groups $\mathcal{A}_8$ and $\mathcal{PSL}(3,4)$ are such that $|\mathcal{A}_8| = 20160 = |\mathcal{PSL}(3,4)|$ and
$\psi(\mathcal{A}_8) = 137047 > 103111 = \psi(\mathcal{PSL}(3,4))$.

# Sum of the orders of the elements - minimum

**Conjecture [H. Amiri, S.M. Jafarian Amiri, 2011]**

Let $G$ be a finite non-simple group, $S$ a finite simple group, $|G| = |S|$.

Then

$$\psi(S) < \psi(G).$$

**Theorem [S.M. Jafarian Amiri, 2013]**

Let $G$ be a finite non-simple group.

If $|G| = 60$, then $\psi(\mathcal{A}_5) < \psi(G)$.

If $|G| = 168$, then $\psi(\mathcal{PSL}(2,7)) < \psi(G)$.

# Sum of the orders of the elements - minimum

**Conjecture [H. Amiri, S.M. Jafarian Amiri, 2011]**

Let $G$ be a finite non-simple group, $S$ a finite simple group, $|G| = |S|$.

Then

$$\psi(S) < \psi(G).$$

**Theorem [S.M. Jafarian Amiri, 2013]**

Let $G$ be a finite non-simple group.

If $|G| = 60$, then $\psi(\mathcal{A}_5) < \psi(G)$.

If $|G| = 168$, then $\psi(\mathcal{PSL}(2,7)) < \psi(G)$.

**Remarks**

Assume $G$ is a finite non-simple group.

Using GAP it is possible to see that:

If $|G| = 360$, then $\psi(\mathcal{A}_6) < \psi(G)$.

If $|G| = 504$, then $\psi(\mathcal{PSL}(2,8)) < \psi(G)$.

If $|G| = 660$, then $\psi(\mathcal{PSL}(2,11)) < \psi(G)$.

If $|G| = 1092$, then $\psi(\mathcal{PSL}(2,13)) < \psi(G)$.

# Sum of the orders of the elements - minimum

But the conjecture is **not** true.

**Theorem [Y. Marefat, A. Iranmanesh, A. Tehranian, 2013]**

Let $S = \mathcal{PSL}(2, 64)$ and $G = 3^2 \times \mathcal{S}z(8)$.

Then $|G| = |S|$ and $\psi(G) \leq \psi(S)$.

**Conjecture**

Let $G$ be a finite soluble group, $S$ a simple group, $|G| = |S|$. Then

$$\psi(S) < \psi(G).$$

# Sum of the orders of the elements - minimum

But the conjecture is **not** true.

**Theorem [Y. Marefat, A. Iranmanesh, A. Tehranian, 2013]**

Let $S = \mathcal{PSL}(2,64)$ and $G = 3^2 \times \mathcal{S}z(8)$.

Then $|G| = |S|$ and $\psi(G) \leq \psi(S)$.

**Conjecture**

Let $G$ be a finite soluble group, $S$ a simple group, $|G| = |S|$. Then

$$\psi(S) < \psi(G).$$

# Sum of the orders of the elements - minimum

But the conjecture is **not** true.

**Theorem [Y. Marefat, A. Iranmanesh, A. Tehranian, 2013]**

Let $S = \mathcal{PSL}(2,64)$ and $G = 3^2 \times \mathcal{S}z(8)$.

Then $|G| = |S|$ and $\psi(G) \leq \psi(S)$.

**Conjecture**

Let $G$ be a finite soluble group, $S$ a simple group, $|G| = |S|$. Then

$$\psi(S) < \psi(G).$$

# Some other functions

Let $G$ be a finite group.

**Definition**

$$\mathcal{P}(G) := \prod_{x \in G} o(x).$$

**Theorem [M. Garonzi, M. Patassini, 2016]**

Let $G$ be a finite group, $|G| = n$. Then

$$\mathcal{P}(G) \leq \mathcal{P}(C_n).$$

Moreover

$$\mathcal{P}(G) = \mathcal{P}(C_n) \text{ if and only if } G \simeq C_n.$$

# Some other functions

Let $G$ be a finite group.

**Definition**

$$\mathcal{P}(G) := \prod_{x \in G} o(x).$$

**Theorem [M. Garonzi, M. Patassini, 2016]**

Let $G$ be a finite group, $|G| = n$. Then

$$\mathcal{P}(G) \leq \mathcal{P}(C_n).$$

Moreover

$$\mathcal{P}(G) = \mathcal{P}(C_n) \text{ if and only if } G \simeq C_n.$$

# Some other functions

Let $G$ be a finite group, $r, s$ real numbers.

**Definition**

$$\mathcal{R}_G(r,s) := \sum_{x \in G} \frac{o(x)^s}{\varphi(o(x))^r}.$$

$$\mathcal{R}_G(r) := \mathcal{R}_G(r,r).$$

**Remark**

$$\mathcal{R}_G(0,1) = \psi(G).$$

# Some other functions

**Theorem [M. Garonzi, M. Patassini, 2016]**

Let $G$ be a finite group, $|G| = n$, $r < 0$.

Then $\mathcal{R}_G(r) \geq \mathcal{R}_{\mathcal{C}_n}(r)$.

Moreover

$\mathcal{R}_G(r) = \mathcal{R}_{\mathcal{C}_n}(r)$ if and only if $G$ is nilpotent.

**Problem**

Let $G$ be a finite group, $|G| = n$, and $r, s$ real numbers.

Does $\mathcal{R}_G(s, r) = \mathcal{R}_{\mathcal{C}_n}(s, r)$ imply $G$ soluble?

**Theorem [T. De Medts, M. Tărnăuceanu, 2008]**

Let $G$ be a finite group, $|G| = n$.

If $G$ is nilpotent, then $\mathcal{R}_G(1) = \mathcal{R}_{\mathcal{C}_n}(1)$.

**Problem**

Let $G$ be a finite group, $|G| = n$.

Does $\mathcal{R}_G(1) = \mathcal{R}_{\mathcal{C}_n}(1)$ imply $G$ nilpotent?

**Problem**

Let $G$ be a finite group, $|G| = n$.

Does $\mathcal{R}_G(1) \leq \mathcal{R}_{\mathcal{C}_n}(1)$?

*Thank you for the attention !*

*Thank you for the attention !*

and again


to **Sandra**

Mercede Maj (P. Longobardi)
Dipartimento di Matematica
Università di Salerno
via Giovanni Paolo II, 132, 84084 Fisciano (Salerno), Italy
E-mail address : mmaj@unisa.it (plongobardi@unisa.it)

# Bibliography


- H. Amiri, S.M. Jafarian Amiri, Sum of element orders on finite groups of the same order, *J. Algebra Appl.* **10** no. 2 (2011), 187-190.
- H. Amiri, S.M. Jafarian Amiri, Sum of element orders of maximal subgroups of the symmetric group, *Comm. Algebra* **40** no. 2 (2012), 770-778.
- H. Amiri, S.M. Jafarian Amiri, I.M. Isaacs, Sums of element orders in finite groups, *Comm. Algebra* **37** (2009), 2978-2980.
- M. Baniasad Asad, B. Khosravi, A Criterion for Solvability of a Finite Group by the Sum of Element Orders, *J. Algebra* **516** (2018), 115-124.

# Bibliography


- A. Bahri, B. Khosravi, Z. Akhlaghi, A result on the sum of element orders of a finite group, arXiv:1904.00425, March 2019.
- M. Garonzi, M. Patassini, Inequalities detecting structural proprieties of a finite group, *Comm. Algebra* **45** (2016), 677-687.
- J. Harrington, L. Jones, A. Lamarche, Characterizing Finite Groups Using the Sum of the Orders of the Elements, *Int. J. Comb.*, Volume 2014, Article ID 835125, 8 pages.
- M. Herzog, P. Longobardi, M. Maj, An exact upper bound for sums of element orders in non-cyclic finite groups, *J. Pure Appl. Algebra*, **222** n. 7 (2018), 1628-1642.

# Bibliography


- M. Herzog, P. Longobardi, M. Maj, Sums of element orders in groups of order $2m$ with $m$ odd, *Comm. Algebra*, to appear.
- M. Herzog, P. Longobardi, M. Maj, Two new criteria for solvability of finite groups in finite groups, *J. Algebra*, **511** (2018), 215-226.
- M. Herzog, P. Longobardi, M. Maj, The second maximal groups with respect to the sum of element orders, submitted, arxiv.org/abs/1901.09662.
- M. Herzog, P. Longobardi, M. Maj, Sums of element orders in groups of odd order, in preparation.

# Bibliography


- M. Herzog, P. Longobardi, M. Maj, Properties of finite and periodic groups determined by their elements orders (a survey), *Group Theory and Computation, Indian Statistical Institute Series*, Editors N.S. Narasimha Sastry, Manoj Kumar Yadav, (2018), Springer, 59-90.
- S.M. Jafarian Amiri, Characterization of $A_5$ and $PSL(2,7)$ by sum of elements orders, *Int. J. Group Theory* **2** (2013), 35-39.
- S.M. Jafarian Amiri, Second maximum sum of element orders on finite nilpotent groups, *Comm. Algebra* **41** no. 6 (2013), 2055-2059.


- S.M. Jafarian Amiri, Characterization of $A_5$ and $PSL(2,7)$ by sum of elements orders, *Int. J. Group Theory* **2** (2013), 35-39.

# Bibliography


- S.M. Jafarian Amiri, Maximum sum of element orders of all proper subgroups of $PGL(2,q)$, *Bull. Iran. Math. Soc.* **39** no. 3 (2013), 501-505.
- S.M. Jafarian Amiri, M. Amiri, Second maximum sum of element orders on finite groups, *J. Pure Appl. Algebra* **218** no. 3 (2014), 531-539.
- S.M. Jafarian Amiri, M. Amiri, Sum of the products of the orders of two distinct elements in finite groups, *Comm. Algebra* **42** no. 12 (2014), 5319-5328.
- S.M. Jafarian Amiri, M. Amiri, Characterization of $p$-groups by sum of the element orders, *Publ. Math. Debrecen* **86** no. 1-2 (2015), 31-37.

# Bibliography


- S.M. Jafarian Amiri, M. Amiri, Sum of the Element Orders in Groups with the Square-Free Order, *Bull. Malays. Math. Sci. Soc.* **40** (2017), 1025-1034.
- Y. Marefat, A. Iranmanesh, A. Tehranian, On the sum of element orders of finite simple groups, *J. Algebra Appl.* **12** no. 7 (2013), 135-138.
- R. Shen, G. Chen, C. Wu, On groups with the second largest value of the sum of element orders, *Comm. Algebra* **43** no. 6 (2015), 2618-2631.
- M. Tărnăuceanu, A criterion for nilpotency of a finite group by the sum of element orders, submitted, arXiv:1903.09744.
- M. Tărnăuceanu, D.G. Fodor, On the sum of element orders of finite abelian groups, *Sci. An. Univ. "A1.I. Cuza" Iasi, Ser. Math.*, **XL** (2014), 1-7.