Amenability in inverse semigroups

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Overview

(1) Introduction (groups)

(2) Inverse semigroups

Definition & examples

Amenability as domain-measurability & localization

(3) Domain-measurable inverse semigroups

Domain-measures as (amenable) traces

(4) Role of the localization

(5) Conclusions & open problems

Joint work with Pere Ara (UAB) and Fernando Lledó (UC3M-ICMAT):

***Amenability in semigroups and C^*-algebras*, ArXiV 1904.13133, 2019.**

(1) Introduction (groups)

Groups and amenability

Theorem-Definitions (von Neumann-Tarski-Følner 19~~)

G countable and discrete group. TFAE:

1. G is amenable, i.e., $\mu : \mathcal{P}(G) \to [0,1]$ normalized such that

$$\mu(A \sqcup B) = \mu(A) + \mu(B) \quad \text{and} \quad \mu\left(g^{-1}A\right) = \mu(A).$$

2. G is not paradoxical: $\nexists\, g_i, h_j \in G, A_i, B_j \subset G$ such that

$$\begin{aligned} G &= g_1 A_1 \sqcup \cdots \sqcup g_n A_n = h_1 B_1 \sqcup \cdots \sqcup h_m B_m \\ &\supset A_1 \sqcup \cdots \sqcup A_n \sqcup B_1 \sqcup \cdots \sqcup B_m. \end{aligned}$$

3. G has a Følner sequence, i.e., $\{F_n\}_{n\in\mathbb{N}}$ with $\varnothing \neq F_n \subset G$ finite

$$|gF_n \cup F_n| / |F_n| \xrightarrow{n\to\infty} 1.$$

(2) Inverse semigroups

Introduction to inverse semigroups I

S *inverse* semigroup:

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Example: $\mathcal{I}(X) = \{(s, A, B) \mid A, B \subset X \text{ and } s\colon A \leftrightarrow B\}$.

- A = domain of $s = D_{s^*s}$.
- B = range of $s = D_{ss^*}$.
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- $e, f \in E(S) \rightsquigarrow \underline{e \leq f \Leftrightarrow ef = e}$.

Example: bicyclic monoid.

$\mathcal{T} := \langle a, a^* \mid a^* a = 1 \rangle$

Introduction to inverse semigroups II

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$$
\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \iddots \\
a^3 & a^3 a^* & a^3 a^{*2} & a^3 a^{*3} & \cdots \\
| & | & | & | & \\
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| & | & | & | & \\
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Introduction to inverse semigroups III

Definition (Day - 1957)

S is amenable if there is an invariant probability measure, i.e., a measure $\mu : \mathcal{P}(S) \to [0,1]$ such that for every $s \in S$ and $A \subset S$

$$\mu\left(s^{-1}A\right) = \mu(A),$$

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Questions: *Alternative* point of view? <u>Amenability *forwards*?</u>

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Proposition (Ara, Lledó, M. - 2019)

μ is invariant $\Leftrightarrow$ both following conditions are satisfied:

1. *Domain-measure*: $\mu(A) = \mu(sA)$ for all $A \subset D_{s^*s}$.
2. *Localization*: $\mu(A) = \mu(A \cap D_{s^*s})$ for all $s \in S, A \subset S$.

Domain-measurable semigroups

Examples of domain-measurable semigroups:

1. All amenable semigroups.
2. Non-inv. & domain-measure: $S = (\{0,1\}, \cdot)$ and $\mu = \delta_1$.
3. Non-ame. & domain-measurable: $S = \mathbb{F}_2 \sqcup \{1\}$ and $\mu = \delta_1$.

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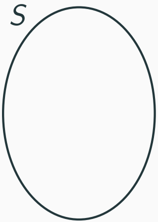

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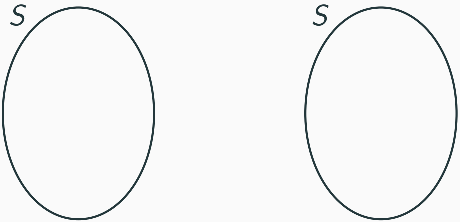

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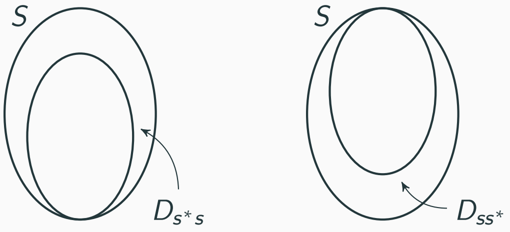

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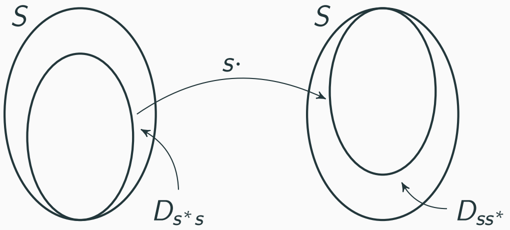

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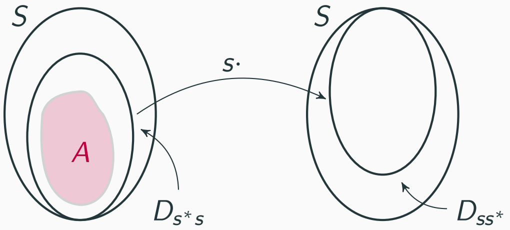

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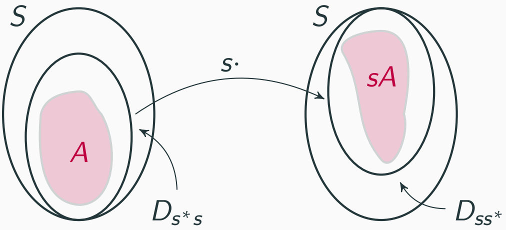

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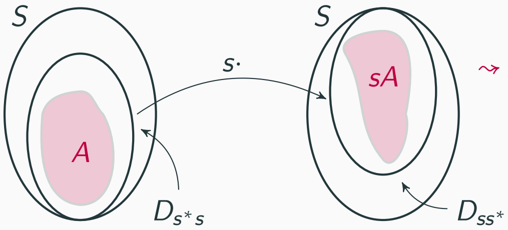

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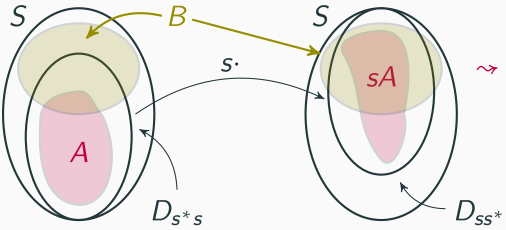

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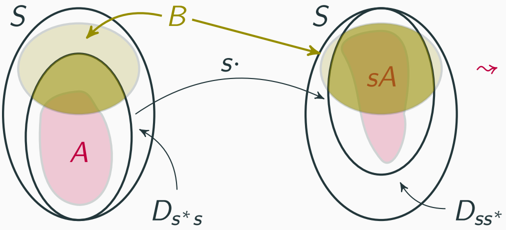

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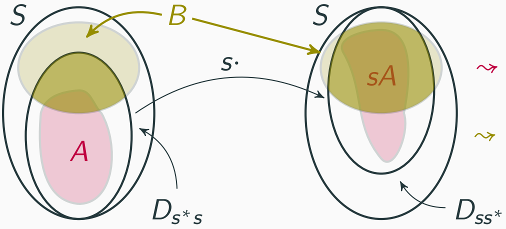

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$= \mu(B \cap D_{ss^*})$.

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Følner sets in domain-measurable semigroups

Følner & paradoxicality $\rightsquigarrow$ *domain point of view*.

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Følner & paradoxicality $\rightsquigarrow$ *domain point of view.*

Definition (Ara, Lledó, M. - 2019)

(i) S is *domain-Følner* if there is $\{F_n\}_{n\in\mathbb{N}}$, with $\varnothing \neq F_n \subset S$ and

$$\frac{|s(F_n \cap D_{s^*s}) \cup F_n|}{|F_n|} \xrightarrow{n\to\infty} 1 \qquad \text{for all } s \in S.$$

(ii) S is *paradoxical* if there are $s_i, t_j \in S$, $A_i \subset D_{s_i^* s_i}$ and $B_j \subset D_{t_j^* t_j}$ with

$$S = s_1 A_1 \sqcup \cdots \sqcup s_n A_n = t_1 B_1 \sqcup \cdots \sqcup t_m B_m$$
$$\supset A_1 \sqcup \ldots A_n \sqcup B_1 \sqcup \cdots \sqcup B_m.$$

Tarski's characterization & domain-measurable

Theorem (Ara, Lledó, M. - 2019)

Let S be countable, discrete & unital. TFAE:

1. S is domain-measurable.
2. S is not paradoxical.
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2 ⇒ 1: Tarski's *type semigroup*.

1 ⇒ 3: extend μ to $\ell^\infty(S)$ & Namioka's trick.

3 ⇒ 1: Consider $\mu(A) := \lim_{n\to\omega} |A \cap F_n| / |F_n|$.

Relation to C^*-algebras:

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→ $\mathcal{R}_S$ can be decomposed into direct sums.

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Theorem (Ara, Lledó, M. - 2019)

S inverse semigroup. Then:

$$\{\text{Traces of } \mathcal{R}_S\} = \{\text{Amenable traces of } \mathcal{R}_S\}$$
$$\leftrightarrow \{\text{Domain-measures of } S\}.$$

(4) Role of the localization

Role of the localization I

Recall: μ invariant $\Leftrightarrow$ μ domain-measure & μ localized.

Theorem (Gray, Kambites - 2017)

S semigroup satisfying the *Klawe condition*. TFAE:

1. S is amenable.
2. S has a *strong* Følner sequence: $\{F_n\}_n$ with $\varnothing \neq F_n \subset S$ and

$$\frac{|sF_n \cup F_n|}{|F_n|} \xrightarrow{n\to\infty} 1 \quad \text{and} \quad |sF_n| = |F_n|.$$

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Localization property: $\mu(B) = \mu(B \cap D_{s^*s})$

$$s\colon D_{s^*s} \mapsto D_{ss^*} \ \underline{\text{injective}} \Rightarrow |F \cap D_{s^*s}| = |s(F \cap D_{s^*s})|.$$

Role of the localization II

Theorem (Ara, Lledó, M. - 2019)

S is amenable iff S has a Følner sequence within the domains:

$$\text{there are } \{F_n\}_{n\in\mathbb{N}} \text{ such that } \varnothing \neq F_n \subset S \text{ and}$$

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$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \iddots \\
a^2 \arrow[d, dash] & a^2 a^* \arrow[d, dash] & a^2 a^{*2} \arrow[d, dash] & a^2 a^{*3} \arrow[d, dash] & \cdots \\
a \arrow[d, dash] & aa^* \arrow[d, dash] & aa^{*2} \arrow[d, dash] & aa^{*3} \arrow[d, dash] & \cdots \\
1 & a^* & a^{*2} & a^{*3} & \cdots
\end{tikzcd}$$

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$$\mathcal{T} = D_1 \qquad \begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \iddots \\ & & & & \\ a^2 & a^2a^* & a^2a^{*2} & a^2a^{*3} & \cdots \\ | & | & | & | & \\ a & aa^* & aa^{*2} & aa^{*3} & \cdots \\ | & | & | & | & \\ 1 & a^* & a^{*2} & a^{*3} & \cdots \end{array}$$

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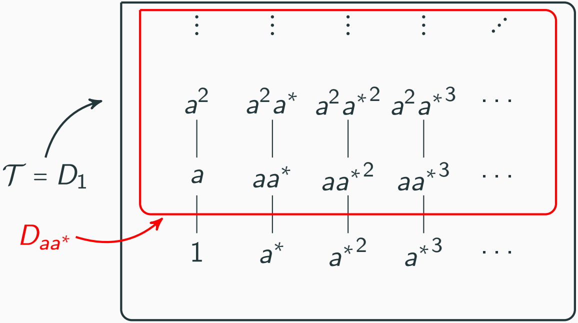

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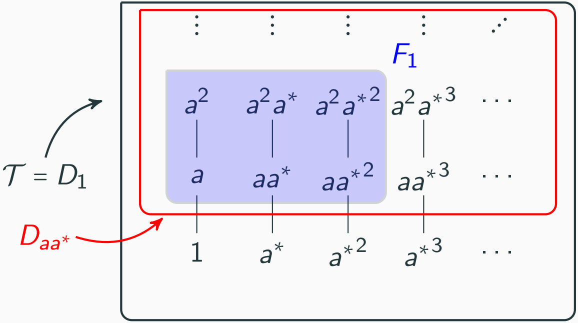

(5) Conclusions & open problems

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Conclusions: amenability behaves well in inverse semigroup theory

... within the domains $D_{s^*s} \subset S$.

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Future research:

1. Do these results have analogues in groupoid theory?
   Within the domains?
2. Relation to other properties?
   - Soficity of S (Ceccherini-Silberstein & Coornaert, 2014).
   - Property A (in metric spaces).
   - Exactness of S.
   - Approximation properties of some operator algebras.

Bibliography & thanks

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Thank you for your attention! Questions?