# Gaps in probabilities of satisfying some commutator identities


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Joint work with

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# Commuting probability

Let $S$ be a finite set with operation $\circ$. Then

$$\mathrm{cp}(S) = \frac{|\{(x, y) \in S^2 \mid x \circ y = y \circ x\}|}{|S|^2}$$

is the probability that a randomly chosen pair of elements in $S$ commutes under $\circ$.

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Question

*If $(S, \circ)$ is non-commutative, how large can $\text{cp}(S)$ be?*

# Commuting probability in semigroups and quasigroups

Semigroups are not exciting; $\mathrm{cp}(S)$ can get arbitrarily close to 1.

## Example (MacHale, 1990)

Let $n \geq 4$, $T_n = \{a_1, a_2, \ldots, a_n\}$, and define

$$a_i \circ a_j = \begin{cases} a_2 & : \quad i = j+1 \\ a_1 & : \quad \text{otherwise} \end{cases}.$$

Then $T_n$ is a semigroup with

$$\mathrm{cp}(T_n) = 1 - 2/n^2.$$

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Then $T_n$ is a semigroup with

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if $S$ is a non-commutative quasigroup, then $\mathrm{cp}(S)$ can get arbitrarily close to 1 (S. M. Buckley, preprint).

# Commuting probability in groups

## Erdös and Turan's description of cp($G$)

### Proposition (Hirsch, 1950; Erdös, Turan, 1968)

*Let $k(G)$ be the number of conjugacy classes in $G$. Then*

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## Erdös and Turan's description of $\mathrm{cp}(G)$

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$$\mathrm{cp}(G) = k(G)/|G|.$$

**Proof.**

Let $x_1, x_2, \ldots, x_k$ be the representatives of conjugacy classes of $G$.

$$\begin{aligned}\mathrm{cp}(G) &= (1/|G|^2) \sum_{x \in G} |C_G(x)| \\ &= (1/|G|^2) \sum_{i=1}^{k} |C_G(x_i)| \cdot |G : C_G(x_i)| \\ &= k(G)/|G|.\end{aligned}$$

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# Commuting probability in groups

## Probability gap

### Theorem (Gustafson, 1973)

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Class equation:

$$\begin{aligned} |G| &= |Z(G)| + \sum_{i=1}^{k(G)-|Z(G)|} |G : C_G(x_i)| \\ &\geq |Z(G)| + 2(k(G) - |Z(G)|). \end{aligned}$$

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There are a lot of further results on the commuting probability of groups, see e.g. Guralnick, Robinson (2006).

# Word maps

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If $G$ is a group, then the map

$$\begin{aligned} w : G^d &\to G \\ (g_1, g_2, \ldots, g_d) &\mapsto w(g_1, g_2, \ldots, g_d) \end{aligned}$$

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## Definition

Let $w = w(x_1, x_2, \ldots, x_d)$ be a word in a free group. A group $G$ is said to **satisfy the identity** $w \equiv 1$ if

$$w(g_1, g_2, \ldots, g_d) = 1$$

for all $g_1, g_2, \ldots, g_d \in G$.

# Word probabilities

## Generalization of the commuting probability

Let $w$ be a word in a free group of rank $d$, and $G$ a finite group. Fix $g \in G$. Then

$$P_{w=g}(G) = \frac{|w^{-1}(g)|}{|G|^d}$$

is the probability that a randomly chosen $d$-tuple of elements of $G$ evaluates to $g$ under the word map $w$.

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Example

$\mathrm{cp}(G) = P_{[x,y]}(G)$.

# Two open problems

Question (Probability Gap Problem; Dixon, 2004)

*Let $w$ be a word. Does there exist $\eta = \eta(w) < 1$ such that every finite group $G$ either satisfies $w \equiv 1$ or $P_w(G) \leq \eta$?*

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Question (Positive Probability Problem; Shalev, 2016)

*Let $w$ be a word and $\epsilon > 0$. Does there exist a word $v = v(w, \epsilon)$ such that every finite group $G$ with $P_w(G) > \epsilon$ satisfies $v \equiv 1$?*

# Example: Commuting probability

## Positive Probability problem for commuting probability

### Theorem (P. M. Neumann, 1989)

*Let $\epsilon > 0$ and let $G$ be a finite group with $\mathrm{cp}(G) > \epsilon$. Then there exist subgroups $N \triangleleft H \triangleleft G$ such that*

- *$H/N$ is abelian,*
- *both $|N|$ and $|G : H|$ are bounded by some function of $\epsilon$.*

*We also say that $G$ is* **($\epsilon$-bounded)-by-abelian-by-($\epsilon$-bounded)**.

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*We also say that $G$ is* **($\epsilon$-bounded)-by-abelian-by-($\epsilon$-bounded)**.

Under the above conditions, there exist integers $m = m(\epsilon)$ and $n = n(\epsilon)$ such that $G$ satisfies the identity

$$[x^m, y^m]^n \equiv 1.$$

# Power words

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- For each $k$ and $d$, there exists a number $0 < p(d,k) < 1$, such that if $d(G) \leq d$ and $P_{x^k}(G) > p(d,k)$, then $G$ satisfies the law $x^k \equiv 1$ (Mann and Martinez, 1996).

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- Let $\epsilon > 0$ and $P_{x^2=a}(G) > \epsilon$ for some $a \in G$. Then $G$ is ($\epsilon$-bounded)-by-abelian-by-($\epsilon$-bounded). (Mann, 2018).

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Theorem

*Let $w$ be either the 2-Engel or the metabelian word. There exists a constant $\delta < 1$ such that whenever $w$ is not an identity in a finite group $G$, we have $P_w(G) \leq \delta$.*

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### Theorem

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The proof strategy outlined here can be (in principle) applied to other words as well.

# A general approach

Projection on the first coordinate; mimic the abelian case.

We may write

$$P_w(G) = \frac{1}{|G|^d} \sum_{g_2,\ldots,g_d \in G} |\{g_1 \in G \mid w(g_1,\ldots,g_d) = 1\}|.$$

Denote

$$C_w(g_2,\ldots,g_d) = \{g_1 \in G \mid w(g_1,\ldots,g_d) = 1\}.$$

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**Caution**: $C_w(g_2,\ldots,g_d)$ is rarely a subgroup in $G$.

# A general approach

## The GOOD and the BAD.

Suppose we define a set $\texttt{BAD} \subseteq G^{d-1}$ with the property that there exist absolute constants $0 < \delta_{\texttt{GOOD}}, \delta_{\texttt{BAD}} < 1$, depending only on $w$ and not on $G$, such that:

1. $\forall (g_2, \ldots, g_d) \in \texttt{GOOD}.\ |C_w(g_2, \ldots, g_d)| \leq \delta_{\texttt{GOOD}} \cdot |G|$
2. $|\texttt{BAD}| \leq \delta_{\texttt{BAD}} \cdot |G|^{d-1}$,

where $\texttt{GOOD} = G^{d-1} \setminus \texttt{BAD}$.

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where $\texttt{GOOD} = G^{d-1} \setminus \texttt{BAD}$.

Then, summing over $\texttt{BAD}$ and $\texttt{GOOD}$ separately, we quickly obtain

$$P_w(G) \leq \delta_{\texttt{GOOD}} + (1 - \delta_{\texttt{GOOD}})\delta_{\texttt{BAD}},$$

which gives an absolute upper bound on the word probability in $G$.

# Sample application: the long commutator

Consider the long commutator word,

$$\gamma_d(x_1, \ldots, x_d) = [x_1, \gamma_{d-1}(x_2, \ldots, x_d)] = [x_1, [x_2, [\ldots, x_d]]].$$

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We know that in a non-abelian group $G$, $P_{\gamma_2}(G) \leq \frac{5}{8}$.

**Theorem**

*Let $G$ be a finite group not satisfying the law $\gamma_d \equiv 1$. Then*

$$P_{\gamma_d}(G) \leq 1 - \frac{3}{2^{d+1}}.$$

*The bound is sharp.*

# Proof of $P_{\gamma_d}(G) \leq 1 - \frac{3}{2^{d+1}}$

$\gamma_d(x_1, \ldots, x_d) = [x_1, \gamma_{d-1}(x_2, \ldots, x_d)]$

Induction on $d$. Let $G$ be a group that does not satisfy the law $\gamma_d \equiv 1$. Set

$$\begin{aligned} \mathtt{BAD} &= \{(g_2, \ldots, g_d) \in G^{d-1} \mid C_{\gamma_d}(g_2, \ldots, g_d) = G\} \\ &= \{(g_2, \ldots, g_d) \in G^{d-1} \mid \gamma_{d-1}(g_2, \ldots, g_d) \in Z(G)\}. \end{aligned}$$

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The size of the latter set is

$$\begin{aligned}|\mathtt{BAD}| = |\{(g_2, \ldots, g_d) \in (G/Z(G))^{d-1} \mid \gamma_{d-1}(g_2, \ldots, g_d) = 1\}| \\ \cdot |Z(G)|^{d-1}.\end{aligned}$$

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$G/Z(G)$ does not satisfy the law $\gamma_{d-1} \equiv 1$. By induction there is a constant $\delta_{d-1}$ with

$$|\{(g_2, \ldots, g_d) \in (G/Z(G))^{d-1} \mid \gamma_{d-1}(g_2, \ldots, g_d) = 1\}| \leq \delta_{d-1}|G/Z(G)|^{d-1}.$$

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so we can take $\delta_{\mathtt{BAD}} = \delta_{d-1}$.

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If $(g_2, \ldots, g_d) \notin \mathtt{BAD}$, we have

$$C_{\gamma_d}(g_2, \ldots, g_d) = C_G(\gamma_{d-1}(g_2, \ldots, g_d)),$$

which is a proper subgroup of $G$, and therefore

$$|C_{\gamma_d}(g_2, \ldots, g_d)| \leq \frac{1}{2}|G|.$$

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This gives a bound for the probability,

$$P_{\gamma_d}(G) \leq \frac{1}{2} + \frac{1}{2}\delta_{d-1} =: \delta_d.$$

Since $\delta_2 = \frac{5}{8}$, we get $\delta_d = 1 - \frac{3}{2^{d+1}}$.

# Central extension of a word with gap

The same argument works more generally.

### Proposition

*Let $w$ be a word in $d$ variables, and suppose that there exists $\eta = \eta(w) < 1$ such that whenever $G$ is a finite group with $P_w(G) > \eta$, then $G$ satisfies the identity $w \equiv 1$.*

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*Let*

$$\tilde{w}(x_1, x_2, \ldots, x_{d+1}) = [x_1, w(x_2, \ldots, x_d)].$$

*Then every finite group $G$ satisfies either $\tilde{w} \equiv 1$ or*

$$P_{\tilde{w}}(G) \leq \frac{1}{2} + \frac{1}{2}\eta(w).$$

## The words $w = [[x_1, x_2], [x_3, x_4]]$ and $w = [x, y, y]$

The naive approach works only in special cases.

For these two words, the approach using projection on a given coordinate works only under extra assumptions on the structure of $G$ (will be elaborated later).

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Alternative strategy is to consider 'minimal counterexamples' and separate the non-solvable case from the solvable case.

It turns out that the non-solvable case can be taken care of using results on word values in simple groups obtained by Bors, Larsen, and Shalev. The solvable case is more *ad hoc*.

# The chief series of a group

### Definition

Let $G$ be a finite group. A **chief series** of $G$ is a chain of normal subgroups of $G$

$$1 = N_0 \leq N_1 \leq N_2 \leq \cdots \leq N_n = G,$$

such that each $N_{i+1}/N_i$ is a minimal normal subgroup of $G/N_i$.

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Each **chief factor** $N_{i+1}/N_i$ is isomorphic to a direct product of isomorphic finite simple groups:

$$N_{i+1}/N_i \cong T^k,$$

where $T$ is a finite simple group.

# First case: non-abelian composition factor

## Reduction

### Lemma

*Let $w$ be a nontrivial word. Let $G$ be a finite group and $N$ a normal subgroup of $G$. Then $P_{w=g}(G) \leq P_{w=gN}(G/N)$ for every $g \in G$.*

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Consider a chief factor $N_2/N_1 = T^k$ of $G$, where $T$ is a simple group and $k \geq 1$. Suppose that $T$ is **non-abelian**.

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In this case, $T$ does not satisfy the identity $w \equiv 1$ (at least in the 2-Engel or metabelian case).

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## Reduction

### Lemma

*Let $w$ be a nontrivial word. Let $G$ be a finite group and $N$ a normal subgroup of $G$. Then $P_{w=g}(G) \leq P_{w=gN}(G/N)$ for every $g \in G$.*

Consider a chief factor $N_2/N_1 = T^k$ of $G$, where $T$ is a simple group and $k \geq 1$. Suppose that $T$ is **non-abelian**.

In this case, $T$ does not satisfy the identity $w \equiv 1$ (at least in the 2-Engel or metabelian case).

For the purposes of our claim, we can replace $G$ by its quotient $G/C_G(N_2/N_1)$ and therefore assume that

$$T^k \leq G \leq \mathrm{Aut}(T^k).$$

# First case: non-abelian composition factor

$T^k \leq G \leq \mathrm{Aut}(T^k)$. When $T$ is large, all is OK

Theorem (Larsen, Shalev, 2017)

*Let $G$ be a finite group such that $T^k \leq G \leq \mathrm{Aut}(T^k)$ for some $k \geq 1$ and a finite nonabelian simple group $T$. Suppose $w$ is a nontrivial word. Then there exist constants $C = C(w)$, $\epsilon = \epsilon(w) > 0$ depending only on $w$ such that, if $|T| \geq C$, then for any $g \in G$ we have $P_{w=g}(G) \leq |T^k|^{-\epsilon}$.*

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As long as $|T| > C$, we therefore have $P_w(G) < C^{-\epsilon}$.

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**Potential problem**: $k$ may not be bounded when $T$ is small.

# First case: non-abelian composition factor

## Multiplicity bounding words

### Definition (Bors, 2017)

A reduced word $w$ is called **multiplicity bounding** if, whenever $G$ is a finite group such that $P_{w=g}(G) > \rho$ for some $g \in G$, the multiplicity of a non-abelian simple group $T$ as a composition factor of $G$ can be bounded above by a function of only $\rho$ and $T$.

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### Proposition

*Let $w \in F_d$ be a multiplicity bounding word. Then there exists a constant $\delta = \delta(w) < 1$ such that every nonsolvable finite group $G$ satisfies $P_w(G) \leq \delta$.*

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### Theorem

*Both the metabelian and 2-Engel word are multiplicity bounding.*

# Second case: solvable groups

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The **verbal subgroup**: $V = \langle w(G^d) \rangle$.

We may assume that all proper quotients of $G$ satisfy $w \equiv 1$, therefore $V$ can be assumed to be the unique minimal normal subgroup of $G$, and so

$$V = \mathbb{F}_p^n.$$

# The metabelian and 2-Engel words

## The projection to the first coordinate works sometimes.

We need a set $\mathtt{BAD} \subseteq G^{d-1}$ and constants $0 < \delta_{\mathtt{GOOD}}, \delta_{\mathtt{BAD}} < 1$, depending only on $w$ and not on $G$, such that:

1. $\forall (g_2, \ldots, g_d) \in \mathtt{GOOD}.\ |C_w(g_2, \ldots, g_d)| \leq \delta_{\mathtt{GOOD}} \cdot |G|$
2. $|\mathtt{BAD}| \leq \delta_{\mathtt{BAD}} \cdot |G|^{d-1}$,

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In our case, this can be done in the following cases:

- $w = [[x_1, x_2], [x_3, x_4]]$: $G'$ acts trivially on $V$.
- $w = [x, y, y]$: $G$ acts trivially on $V$.

# Solvable groups: Non-trivial action

General principle: `GOOD` and `BAD` representatives

Let $\mathcal{R}$ be a set of coset representatives for $V$ in $G$. Then

$$P_w(G) = \frac{1}{|G|^d} \sum_{a_i \in V, r_i \in \mathcal{R}} \mathbb{1}_{w(a_1 r_1, \ldots, a_d r_d) = 1}.$$

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Each summand can be expanded as

$$w(a_1 r_1, \dots, a_d r_d) = \prod_{i=1}^{d} a_i^{w_i(r_1, \dots, r_d)} \cdot w(r_1, \dots, r_d)$$

for some endomorphisms $w_i(r_1, \dots, r_d) \in \mathrm{End}(V)$.

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for some endomorphisms $w_i(r_1, \ldots, r_d) \in \mathrm{End}(V)$.
Set

$$\begin{aligned} \mathtt{BAD} &= \{(r_1, \ldots, r_d) \in \mathcal{R}^d \mid \forall i.\ w_i(r_1, \ldots, r_d) = 0_{\mathrm{End}(V)}\}, \\ \mathtt{GOOD} &= \mathcal{R}^d - \mathtt{BAD}. \end{aligned}$$

# Solvable groups: Non-trivial action

## Summation over the bad representatives

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BAD consists of those tuples of elements of $\mathcal{R}$ for which a summand above is independent of the values $a_i \in V$.

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By first summing over the bad representatives, we have

$$\frac{1}{|G|^d} \sum_{(r_1, \ldots, r_d) \in \mathtt{BAD}} |V|^d \cdot \mathbb{1}_{w(r_1, \ldots, r_d) = 1} \leq \frac{|\mathtt{BAD}|}{|\mathcal{R}|^d}.$$

# Solvable groups: Non-trivial action

## Summation over the good representatives

$$w(a_1 r_1, \ldots, a_d r_d) = \prod_{i=1}^{d} a_i^{w_i(r_1,\ldots,r_d)} \cdot w(r_1, \ldots, r_d),$$

$$\mathtt{GOOD} = \{(r_1, \ldots, r_d) \in \mathcal{R}^d \mid \exists i.\ w_i(r_1, \ldots, r_d) \neq 0_{\mathsf{End}(V)}\},$$

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If $(r_1, \ldots, r_d) \in \mathtt{GOOD}$, then there is $j$ such that

$$|C_V(w_j(r_1, \ldots, r_d))| \leq |V|/p.$$

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Summing over the good representatives, we have

$$\frac{1}{|G|^d} \sum_{(r_1,\ldots,r_d)\in\mathtt{GOOD}} \sum_{a_i \in V} \mathbb{1}_{w(a_1 r_1,\ldots,a_d r_d)=1} \leq \frac{1}{|G|^d} \sum_{(r_1,\ldots,r_d)\in\mathtt{GOOD}} |V|^{d-1}(|V|/p)$$
$$= \frac{|\mathtt{GOOD}|}{p|\mathcal{R}|^d}$$

# Solvable groups: Non-trivial action

## Putting `BAD` and `GOOD` together

We can collect the two upper bounds to finally obtain

$$\begin{aligned} P_w(G) &\leq \frac{|\mathtt{BAD}|}{|\mathcal{R}|^d} + \frac{|\mathtt{GOOD}|}{p|\mathcal{R}|^d} \\ &= \frac{1}{p} + \left(1 - \frac{1}{p}\right) \frac{|\mathtt{BAD}|}{|\mathcal{R}|^d} \\ &\leq \frac{1}{2}\left(1 + \frac{|\mathtt{BAD}|}{|\mathcal{R}|^d}\right). \end{aligned}$$

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\end{aligned}
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In order to get a gap on word probability, we need to show that for a given word $w$, there is a gap on the relative size of the set BAD inside $\mathcal{R}^d$.

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Metabelian and 2-Engel words. Breaking BAD: the GOOD, the BAD and the UGLY

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In the metabelian case we need to assume $[G, G]$ acts non-trivially on $V$, i.e., $G' \not\subseteq C_G(V)$. Define

$$\mathtt{UGLY} = \{(z, t) \in (G/V)^2 \mid [z, t] \in C_G(V)\}.$$

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Then we can show that

$$\frac{|\mathtt{BAD}|}{|\mathcal{R}|^4} \leq \frac{1}{2} + \frac{1}{2}\frac{|\mathtt{UGLY}|}{|\mathcal{R}|^2} \quad \text{and} \quad \frac{|\mathtt{UGLY}|}{|\mathcal{R}|^2} \leq \frac{5}{8}.$$