Groups with many self-centralizing or self-normalizing subgroups

Chiara Nicotera

Cremona, 12 giugno 2019

References

- C. Delizia, U. Jezernik, P. Moravec, C. Nicotera, *Groups in which every non-cyclic subgroup contains its centralizer*, J. Algebra Appl., **13** no.5 (2014).
- C. Delizia, H. Dietrich, P. Moravec and C. Nicotera, *Groups in which every non-abelian subgroup is self-centralizing*, J. Algebra, **462** (2016) 23-36.
- C. Delizia, U. Jezernik, P. Moravec, C. Nicotera and C.Parker, *Locally finite groups in which every non-cyclic subgroups is self-centralizing*, J. Pure Appl. Algebra, **221** no.2 (2017) 401-410.
- C. Delizia, U. Jezernik, P. Moravec, C. Nicotera, *Groups in which every non-abelian subgroup is self-normalizing*, Monatsh. Math., **185** (2018) 591-600.
- C. Delizia, U. Jezernik, P. Moravec, C. Nicotera , *Groups in which every non-nilpotent subgroup is self-normalizing*, Ars Math. Contemp., **15** no.1 (2018) 39-51.

Let H be a subgroup of a group G

- $C_G(H) = \{g \in G | gh = hg \ \forall h \in H\}$ centralizer of H in G
- $N_G(H) = \{g \in G | H^g = H\}$ normalizer of H in G

Of course $C_G(H) \subseteq N_G(H)$, $H \subseteq N_G(H)$

Definition

A subgroup H of a group G is **self-centralizing** if $C_G(H) \subseteq H$

A non-abelian group G has always proper self-centralizing subgroups

The subgroup $\{1\}$ is self-centralizing iff $G = \{1\}$

The following subgroups of a group G are self-centralizing:

- *maximal abelian subgroups*
- *the Fitting subgroup $F(G)$ of a soluble group*
- *the generalized Fitting subgroup $F^*(G)$ of a group G*

Definition

A subgroup H of a group G is **self-normalizing** if $N_G(H) = H$

If H is self-normalizing, then it is self-centralizing since $C_G(H) \subseteq N_G(H)$

Remark

Every self-normalizing subgroup is self- centralizing. For a finite group G, the converse holds if and only if G is abelian (M. Hassanzadeh, Z. Mostaghim 2018)

The following subgroups of a finite group G are self-normalizing:

- *the normalizer* $N_G(P)$ *of any p-Sylow subgroup*
- *Carter subgroups*
- *the complement of any Frobenius group*

Let $\mathfrak{X}$ be a class of groups and consider

- $\mathfrak{C}_{\mathfrak{X}}$ groups in which every non-$\mathfrak{X}$ subgroup is self-centralizing
- $\mathfrak{N}_{\mathfrak{X}}$ groups in which every non-$\mathfrak{X}$ subgroup is self-normalizing

of course $\mathfrak{N}_{\mathfrak{X}} \subseteq \mathfrak{C}_{\mathfrak{X}}$

The "smallest" class

If $G \in \mathfrak{N}_{tr}$ is a group in which every non-trivial subgroup is self-normalizing, then

- $G \in \mathfrak{N}_{\mathfrak{X}}$ for every $\mathfrak{X}$ class of groups.
- $G \in \mathfrak{C}_{tr}$ is a group in which every non-trivial subgroup is self-centralizing, then $G \in \mathfrak{C}_{\mathfrak{X}}$ for every $\mathfrak{X}$ class of groups.

Every Tarski p-group lies in the class $\mathfrak{N}_{tr}$ and so such groups belong to each class $\mathfrak{N}_{\mathfrak{X}}, \mathfrak{C}_{\mathfrak{X}}$.

Notice

We restrict our investigation to locally graded groups

The classes $\mathfrak{C}_{tr}$ and $\mathfrak{N}_{tr}$

Groups in which every **non-trivial** subgroup is self-centralizing, groups in which every **non-trivial** subgroup is self-normalizing

Remark

$\mathfrak{N}_{tr} \subset \mathfrak{C}_{tr}$ the inclusion is proper: $S_3 \in \mathfrak{C}_{tr} \setminus \mathfrak{N}_{tr}$

Theorem

Let G be a locally graded $\mathfrak{C}_{tr}$-group. Then G is finite.

Theorem

A finite group G lies in the class $\mathfrak{C}_{tr}$ iff one of the following holds:

1. $G \cong C_p$
2. *G is non-abelian of order pq where $p \neq q$ are primes*

Corollary

A locally graded group G lies in the class $\mathfrak{N}_{tr}$ iff $G \cong C_p$.

The classes $\mathfrak{C}_c$ and $\mathfrak{N}_c$

Groups in which every **non-cyclic** subgroup is self-centralizing, groups in which every **non-cyclic** subgroup is self-normalizing

Remark

$\mathfrak{N}_c \subset \mathfrak{C}_c$ the inclusion is proper: $D_8 \in \mathfrak{C}_c \setminus \mathfrak{N}_c$

Every cyclic group lies in the class $\mathfrak{C}_c$

$\mathfrak{C}_{tr} \subset \mathfrak{C}_c$

Theorem

Classification of all finite groups in the class $\mathfrak{C}_c$

The class $\mathfrak{C}_c$

Groups in which every **non-cyclic** subgroup is self-centralizing

The classification of all finite groups in the class $\mathfrak{C}_c$ **1/5**

Let G be a finite $\mathfrak{C}_c$-group. Then one of the following holds:

(1) If G is nilpotent, then either

(1.1) G is cyclic;
(1.2) G is elementary abelian of order p^2 for some prime p;
(1.3) G is an extraspecial p-group of order p^3 for some odd prime p; or
(1.4) G is a dihedral, semidihedral or quaternion 2-group.

The class $\mathfrak{C}_c$

Groups in which every **non-cyclic** subgroup is self-centralizing

The classification of all finite groups in the class $\mathfrak{C}_c$ **2/5**

(2) If G is supersoluble but not nilpotent, then, letting p denote the largest prime divisor of $|G|$ and $P \in \mathrm{Syl}_p(G)$, we have that P is a normal subgroup of G and one of the following holds:

- (2.1) P is cyclic and either
  - (2.1.1) $G = C \rtimes D$, where C is cyclic, D is cyclic and D acts fixed point freely on C (so G is a Frobenius group);
  - (2.1.2) $G = C \rtimes D$, where C is a cyclic group of odd order, D is a quaternion group, and $C_G(C) = C \times D_0$ where D_0 is a cyclic subgroup of index 2 in D; or
  - (2.1.3) $G = C \rtimes D$, where D is a cyclic q-group, C is a cyclic q'-group (here q denotes the smallest prime dividing the order of G) and $1 < Z(G) < D$;
- (2.2) P is extraspecial and G is a Frobenius group with cyclic Frobenius complement of odd order dividing $p - 1$.

The class $\mathfrak{C}_c$

Groups in which every **non-cyclic** subgroup is self-centralizing

The classification of all finite groups in the class $\mathfrak{C}_c$ **3/5**

(3) If G is not supersoluble and $F^*(G)$ is elementary abelian of order p^2, then $F^*(G)$ is a minimal normal subgroup of G and one of the following holds:

(3.1) $G \cong \mathrm{Sym}(4)$ or $G \cong \mathrm{Alt}(4)$; or

(3.2) p is odd and $G = N \rtimes G_0$ is a Frobenius group with Frobenius kernel N and Frobenius complement G_0. Furthermore, either

(3.2.1) G_0 is cyclic of order dividing $p^2 - 1$ but not dividing $p - 1$;

(3.2.2) G_0 is quaternion;

(3.2.3) G_0 is supersoluble as in (2.1.2) with $|C|$ dividing $p - \epsilon$ if $p \equiv \epsilon \pmod 4$;

(3.2.4) G_0 is supersoluble as in (2.1.3) with D a 2-group, $C_D(C)$ a non-trivial maximal subgroup of D and $|C|$ odd dividing $p^2 - 1$;

(3.2.5) $G_0 \cong \mathrm{SL}_2(3)$;

(3.2.6) $G_0 \cong \mathrm{SL}_2(3)^{\cdot}2$ if $p \equiv \pm 1 \pmod 8$; or

(3.2.7) $G_0 \cong \mathrm{SL}_2(5)$ if 60 divides $p^2 - 1$.

The class $\mathfrak{C}_c$

Groups in which every **non-cyclic** subgroup is self-centralizing

The classification of all finite groups in the class $\mathfrak{C}_c$ **4/5**

(4) If G is not supersoluble and $F^*(G)$ is extraspecial of order p^3, then one of the following holds:

(4.1) $G \cong \mathrm{SL}_2(3)$ or $G \cong \mathrm{SL}_2(3){\cdot}2$ (with quaternion Sylow 2-subgroups of order 16); or

(4.2) $G = N \rtimes K$ where N is extraspecial of order p^3 and exponent p with p an odd prime, K centralizes $Z(N)$ and is cyclic of odd order dividing $p+1$. Furthermore, $G/Z(N)$ is a Frobenius group.

The class $\mathfrak{C}_c$

Groups in which every **non-cyclic** subgroup is self-centralizing

The classification of all finite groups in the class $\mathfrak{C}_c$ **5/5**

(5) If $F^*(G)$ is not nilpotent, then either

- (5.1) $F^*(G) \cong \mathrm{SL}_2(p)$ where p is a Fermat prime, $|G/F^*(G)| \leq 2$ and G has quaternion Sylow 2-subgroups; or
- (5.2) $G \cong \mathrm{PSL}_2(9)$, $\mathrm{Mat}(10)$ or $\mathrm{PSL}_2(p)$ where p is a Fermat or Mersenne prime.

Furthermore, all the groups listed above are $\mathfrak{C}_c$-groups.

The class $\mathfrak{C}_c$

Groups in which every **non-cyclic** subgroup is self-centralizing

Theorem

- *An infinite abelian group lies in $\mathfrak{C}_c$ iff it is either cyclic or isomorphic to $\mathbb{Z}_{p^\infty}$ for some prime p.*
- *Every infinite nilpotent group in the class $\mathfrak{C}_c$ is abelian.*
- *An infinite polycyclic group lies in $\mathfrak{C}_c$ if and only if it is either cyclic or dihedral.*

The class $\mathfrak{C}_c$

Groups in which every **non-cyclic** subgroup is self-centralizing

Theorem

Let G be an infinite soluble group. Then G is a $\mathfrak{C}_c$-group if and only if one of the following holds:

1. *G is cyclic;*
2. *G is dihedral;*
3. $G \cong \mathbb{Z}_{p^\infty}$ *for some prime p;*
4. $G = A\langle y\rangle$ *where* $A \cong \mathbb{Z}_{2^\infty}$ *and* $\langle y\rangle$ *has order* 2 *or* 4*, with* $y^2 \in A$ *and* $a^y = a^{-1}$*, for all* $a \in A$*;*
5. $G \cong A \rtimes D$*, where* $A \cong \mathbb{Z}_{p^\infty}$ *and* $\{1\} \neq D \leq C_{p-1}$ *for some odd prime p.*

Infinite locally finite $\mathfrak{C}_c$*-groups are precisely groups in* $3 - 4 - 5$ *above*

Infinite locally nilpotent $\mathfrak{C}_c$*-groups are precisely groups in* $1 - 3 - 4$ *above*

The class $\mathfrak{C}_c$

Groups in which every **non-cyclic** subgroup is self-centralizing

Residually finite $\mathfrak{C}_c$-groups

- Free groups are examples of torsion-free residually finite $\mathfrak{C}_c$-groups
- D_∞ is an example of of infinite residually finite $\mathfrak{C}_c$-group having periodic elements
- Do exist infinite periodic residually finite $\mathfrak{C}_c$-groups?

C. Delizia, C.N. 2019

Let G be an infinite perodic, residually finite $\mathfrak{C}_c$-group. Then:

- the centralizer $C_G(a)$ is finite for all $1 \neq a \in G$;
- every non trivial normal subgroup of G is infinite;
- the centre $Z(G) = \{1\}$;
- the Fitting subgroup $F(G) = \{1\}$;
- if H is an infinte subgroup of G, then $C_G(H) = \{1\}$;
- G has no element of order 2.

The class $\mathfrak{N}_c$

Groups in which every **non-cyclic** subgroup is self-normalizing

Theorem (C. Delizia, C.N. 2019)

A finite group G lies in the class $\mathfrak{N}_c$ if and only if one of the following holds:

- *G is cyclic ;*
- $G \cong C_p \times C_p$
- $G \cong Q_8$;
- *G is a Frobenius group with cyclic kernel of odd order and complement of prime order;*
- $G \cong D \ltimes C$, *where D is a cyclic q-group (q being the smallest prime dividing the order of G), C is a cyclic q'-group, $\{1\} < Z(G) < D$, $|\frac{D}{Z(G)}| = q$ and $\frac{G}{Z(G)}$ is a Frobenius group.*

Theorem (C. Delizia, C.N. 2019)

Every infinite locally graded $\mathfrak{N}_c$-group is abelian.

The classes $\mathfrak{C}_{ab}$ and $\mathfrak{N}_{ab}$

Groups in which every **non-abelian** subgroup is self-centralizing, groups in which every **non-abelian** subgroup is self-normalizing

Remark

$\mathfrak{N}_{ab} \subset \mathfrak{C}_{ab}$ the inclusion is proper: $S_4 \in \mathfrak{C}_{ab} \setminus \mathfrak{N}_{ab}$

The class $\mathfrak{C}_{ab}$

Groups in which every **non-abelian** subgroup is self-centralizing

Every abelian group lies in the class $\mathfrak{C}_{ab}$

$\mathfrak{C}_{tr} \subset \mathfrak{C}_c \subset \mathfrak{C}_{ab}$ *the wreath product* $C_3 \wr C_3 \in \mathfrak{C}_{ab}$ *but* $C_3 \wr C_3 \notin \mathfrak{C}_c$

The class $\mathfrak{C}_{ab}$ also contains:

1. **minimal non-abelian groups** *(non-abelian groups in which every proper subgroup is abelian)*
2. **commutative-transitive groups** *(groups in which the centralizer of each non-trivial element is abelian)*

The class $\mathfrak{C}_{ab}$

Groups in which every **non-abelian** subgroup is self-centralizing

Theorem

Every nilpotent $\mathfrak{C}_{ab}$-group G is either abelian or a finite p-group.

Open Problem (Y. Berkovich, 2008)

Determine all finite p-groups lying in the class $\mathfrak{C}_{ab}$

Theorem

Every finite metacyclic p-group lies in the class $\mathfrak{C}_{ab}$

Theorem

Let G be a p-group of maximal class having order p^n.

1. *If $p \in \{2, 3\}$ or $n \leq 3$, then $G \in \mathfrak{C}_{ab}$;*
2. *If $p \geq 5$ and $n \geq 4$, then G is a $\mathfrak{C}_{ab}$-group if and only if $C_G(\gamma_2(G)/\gamma_4(G))$ is abelian.*

The class $\mathfrak{C}_{ab}$

Groups in which every **non-abelian** subgroup is self-centralizing

Theorem

Let $G \in \mathfrak{C}_{ab}$ be a finite p-group of exponent p. If $|G| > p^p$, then G is elementary abelian. Otherwise, either G is elementary abelian or G has maximal class and an elementary abelian subgroup of index p.

Theorem

Let G be an infinite supersoluble group.

- *If G has no elements of even order and it is a $\mathfrak{C}_{ab}$-group, then G is abelian.*
- *If G is not abelian, then G is a $\mathfrak{C}_{ab}$-group if and only if $G = A\langle x\rangle$ where A is abelian, $a^x = a^{-1}$ for all $a \in A$, $|x| = 2^n$, $A = \langle a_1\rangle \times \cdots \times \langle a_t\rangle \times \langle d\rangle$, a_i of infinite or odd order for all $i \in \{1, \ldots, t\}$, $|d| = 2^h$, $d^{2^{h-1}} = x^{2^{n-1}}$ if $h > 0$.*

The class $\mathfrak{N}_{ab}$

Groups in which every **non-abelian** subgroup is self-normalizing

Minimal non-abelian groups lie in the class $\mathfrak{N}_{ab}$

$\mathfrak{N}_{tr} \subset \mathfrak{N}_c \subset \mathfrak{N}_{ab}$ *the elementary abelian* 2*-group* $C_2 \times C_2 \times C_2 \in \mathfrak{N}_{ab} \setminus \mathfrak{N}_c$

Theorem

Every nilpotent $\mathfrak{N}_{ab}$*-group is either abelian or a finite minimal non-abelian* p*-group for some prime* p*.*

The class $\mathfrak{N}_{ab}$

Groups in which every **non-abelian** subgroup is self-normalizing

Proposition

Let $G \in \mathfrak{N}_{ab}$ be a finite group; then G is either soluble or simple.

Theorem

Let G be a finite group.

- *If G is a non-abelian simple group, then $G \in \mathfrak{N}_{ab}$ iff $G \simeq \mathrm{Alt}(5)$ or $G \simeq \mathrm{PSL}_2(2^{2n+1})$, $n \geq 1$.*
- *If G is a soluble non-nilpotent group, then $G \in \mathfrak{N}_{ab}$ iff $G = A \rtimes \langle x \rangle$, where $\langle x \rangle$ is a p-group for some prime p, A is an abelian p'-group, x^p is central and x acts fixed point freely on A.*
- *If G is a nilpotent group, then $G \in \mathfrak{N}_{ab}$ iff G is either abelian or minimal non-abelian p-group for some prime p*

The class $\mathfrak{N}_{ab}$

Groups in which every **non-abelian** subgroup is self-normalizing

Theorem

Let $G \in \mathfrak{N}_{ab}$ if G is non-periodic and soluble, then G is abelian.

Theorem

Let G be an infinite periodic group and suppose that G is soluble and $dl(G) > 1$. Then $G \in \mathfrak{N}_{ab}$ iff $G = \langle x \rangle \ltimes G'$, where x is an element of prime power order p^n, $x^p \in C_G(G')$ and G' is an abelian group with no elements of order p.

Theorem

If G is a locally finite $\mathfrak{N}_{ab}$-group, then it is either finite or soluble.

The class $\mathfrak{N}_{nil}$

Groups in which every **non-nilpotent** subgroup is self-normalizing

Nilpotent groups belong to $\mathfrak{N}_{nil}$ and minimal non-nilpotent groups lie in the class $\mathfrak{N}_{nil}$

$\mathfrak{N}_{tr} \subset \mathfrak{N}_c \subset \mathfrak{N}_{ab} \subset \mathfrak{N}_{nil}$, *also the last inclusion is proper since* $SL_2(5) \in \mathfrak{N}_{nil} \setminus \mathfrak{N}_{ab}$

Remark

Let G be a $\mathfrak{N}_{nil}$-group; then

- either $G = G'$
- or G' is nilpotent (and so G is soluble)

Soluble $\mathfrak{N}_{nil}$-groups

Soluble groups in which every **non-nilpotent** subgroup is self-normalizing

Theorem

Let G be a soluble non-periodic group; then G is a $\mathfrak{N}_{nil}$-group iff G is nilpotent

Theorem

Let G be a periodic soluble group, and suppose G is not locally nilpotent; then G is a $\mathfrak{N}_{nil}$-group iff

- *$G = H \rtimes \langle x \rangle$, where $\langle x \rangle$ is a p-group for some prime p, H is a nilpotent p'-group and x^p acts trivially on H;*
- *put $\rho_x : h \in H \to h^{-x}h \in H$, for every $\langle x \rangle$-invariant subgroup K of H either there exists $n \geq 1$ such that $\rho_x^n(K) = 1$, or $\langle \rho_x(K) \rangle = K$.*

Theorem

Let G be a soluble, locally nilpotent group; then G is a $\mathfrak{N}_{nil}$-group iff either it is nilpotent or it is minimal non-nilpotent and it is a p-group for some prime p.

Perfect $\mathfrak{N}_{nil}$-groups

Perfect groups in which every **non-nilpotent** subgroup is self-normalizing

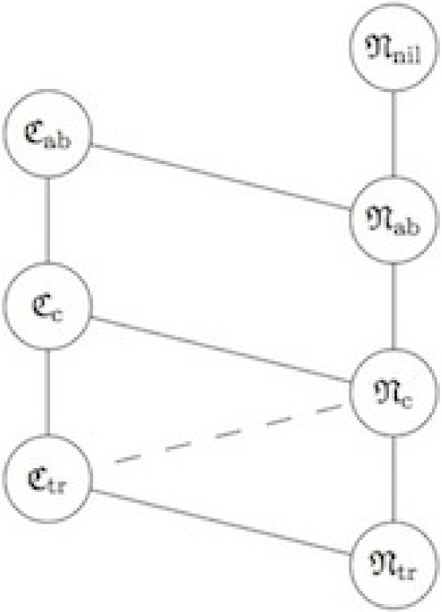
Theorem

Let G be a finite perfect group; then G is a $\mathfrak{N}_{nil}$-group iff

- *either $G \simeq PSL_2(2^n)$, where $2^n - 1$ is a prime*
- *or $G \simeq SL_2(5)$*

Lemma

If G is a perfect $\mathfrak{N}_{nil}$-group and $F := F(G)$ is its Fitting subgroup, then G/F is a non-abelian simple group.


$\mathfrak{N}_{nil}$
$\mathfrak{C}_{ab}$
$\mathfrak{N}_{ab}$
$\mathfrak{C}_{c}$
$\mathfrak{N}_{c}$
$\mathfrak{C}_{tr}$
$\mathfrak{N}_{tr}$