Engel elements in groups of automorphisms of rooted trees

Marialaura Noce
(joint with Fernández-Alcober, Garreta, Tortora, and Tracey)

University of Salerno - University of the Basque Country

June 2019 - Cremona
SandGAL 2019

Outline

Engel elements

- Let G be a group. We say that $g \in G$ is a *left Engel element* if for any $x \in G$, $\exists n = n(g,x) \geq 1$ such that $[x, g, \overset{n}{\ldots}, g] = 1$.

Engel elements

- Let G be a group. We say that $g \in G$ is a *left Engel element* if for any $x \in G$, $\exists n = n(g,x) \geq 1$ such that $[x, g, \overset{n}{\ldots}, g] = 1$.
- If n can be chosen independently of x, then g is a *bounded left Engel element*.

Engel elements

- Let G be a group. We say that $g \in G$ is a *left Engel element* if for any $x \in G$, $\exists n = n(g, x) \geq 1$ such that $[x, g, \overset{n}{\ldots}, g] = 1$.
- If n can be chosen independently of x, then g is a *bounded left Engel element*.
- Notation:
  - $L(G) = \{$left Engel elements of $G\}$
  - $\bar{L}(G) = \{$bounded left Engel elements of $G\}$

Engel elements

- Let G be a group. We say that $g \in G$ is a *left Engel element* if for any $x \in G$, $\exists n = n(g,x) \geq 1$ such that $[x, g, \overset{n}{\ldots}, g] = 1$.
- If n can be chosen independently of x, then g is a *bounded left Engel element*.
- Notation:
  - $L(G) = \{$left Engel elements of $G\}$
  - $\bar{L}(G) = \{$bounded left Engel elements of $G\}$
- Similar considerations for right Engel elements.

Engel elements

- Let G be a group. We say that $g \in G$ is a *left Engel element* if for any $x \in G$, $\exists n = n(g,x) \geq 1$ such that $[x, g, \overset{n}{\ldots}, g] = 1$.
- If n can be chosen independently of x, then g is a *bounded left Engel element*.
- Notation:
  - $L(G) = \{$left Engel elements of $G\}$
  - $\bar{L}(G) = \{$bounded left Engel elements of $G\}$
- Similar considerations for right Engel elements.
  We have the sets $R(G)$ and $\bar{R}(G)$.

Engel elements

- Let G be a group. We say that $g \in G$ is a *left Engel element* if for any $x \in G$, $\exists n = n(g,x) \geq 1$ such that $[x, g, \overset{n}{\ldots}, g] = 1$.
- If n can be chosen independently of x, then g is a *bounded left Engel element*.
- Notation:
  - $L(G) = \{\text{left Engel elements of } G\}$
  - $\bar{L}(G) = \{\text{bounded left Engel elements of } G\}$
- Similar considerations for right Engel elements.
  We have the sets $R(G)$ and $\bar{R}(G)$.
- If $L(G) = G$ or $R(G) = G$, then G is an *Engel group*.

An Engel version of the Burnside problem

An Engel version of the Burnside problem

Golod-Shafarevich groups

An Engel version of the Burnside problem

Golod-Shafarevich groups and many periodic subgroups of $\operatorname{Aut}\mathcal{T}_d$ (the group of automorphisms of a rooted tree) answer the General Burnside problem negatively.

An Engel version of the Burnside problem

Golod-Shafarevich groups and many periodic subgroups of $\operatorname{Aut} \mathcal{T}_d$ (the group of automorphisms of a rooted tree) answer the General Burnside problem negatively.

Question 1: if G is Engel, is G locally nilpotent? (Burnside, 1902)

An Engel version of the Burnside problem

Golod-Shafarevich groups and many periodic subgroups of Aut $\mathcal{T}_d$ (the group of automorphisms of a rooted tree) answer the General Burnside problem negatively.

Question 1: if G is Engel, is G locally nilpotent? (Burnside, 1902)

- Positive: finite groups (Zorn, 1936), solvable groups (Gruenberg, 1959), compact groups (Medvedev, 2003), and many others ...

An Engel version of the Burnside problem

Golod-Shafarevich groups and many periodic subgroups of Aut $\mathcal{T}_d$ (the group of automorphisms of a rooted tree) answer the General Burnside problem negatively.

Question 1: if G is Engel, is G locally nilpotent? (Burnside, 1902)

- Positive: finite groups (Zorn, 1936), solvable groups (Gruenberg, 1959), compact groups (Medvedev, 2003), and many others ...
- Negative: some Golod-Shafarevich groups (1964).

An Engel version of the Burnside problem

Golod-Shafarevich groups and many periodic subgroups of $\mathrm{Aut}\,\mathcal{T}_d$ (the group of automorphisms of a rooted tree) answer the General Burnside problem negatively.

Question 1: if G is Engel, is G locally nilpotent? (Burnside, 1902)

- Positive: finite groups (Zorn, 1936), solvable groups (Gruenberg, 1959), compact groups (Medvedev, 2003), and many others ...
- Negative: some Golod-Shafarevich groups (1964).

Does any subgroup of $\mathrm{Aut}\,\mathcal{T}_d$ solve also the Engel Burnside problem?

An Engel version of the Burnside problem

Question 2: are $L(G), \bar{L}(G), \ldots$ subgroups of a given group G?

An Engel version of the Burnside problem

Question 2: are $L(G), \bar{L}(G), \dots$ subgroups of a given group G?

For $L(G)$:

- Negative: the first Grigorchuk group (Bartholdi, 2016).

An Engel version of the Burnside problem

Question 2: are $L(G), \bar{L}(G), \dots$ subgroups of a given group G?

For $L(G)$:

- Negative: the first Grigorchuk group (Bartholdi, 2016).

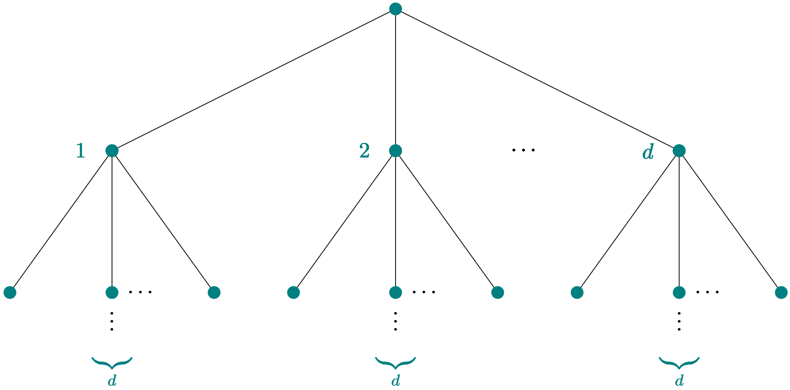
For $\bar{L}(G), R(G), \bar{R}(G)$:

- Open.

Outline

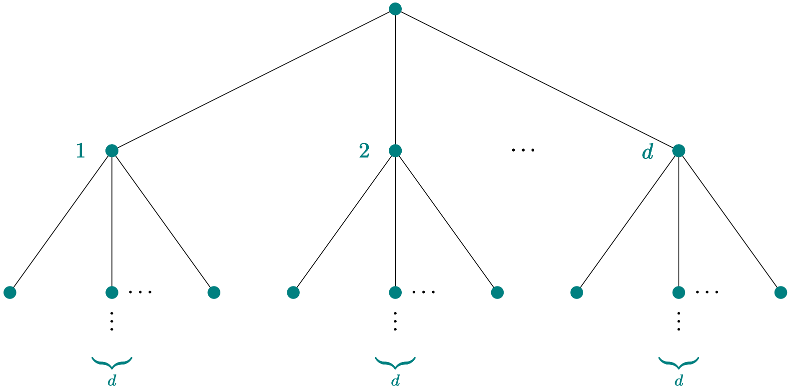
Automorphisms of a d-adic tree

- Regular rooted trees $\mathcal{T}_d$



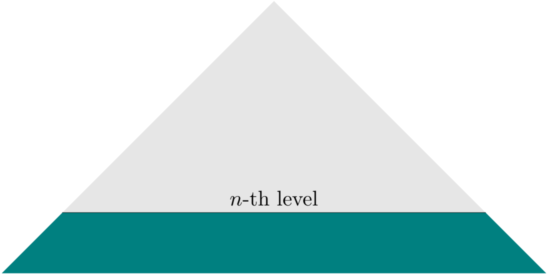
Automorphisms of a d-adic tree

- Regular rooted trees $\mathcal{T}_d$



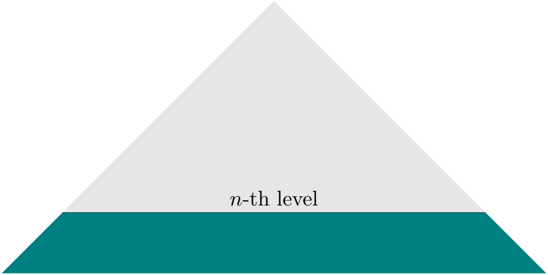
- Automorphisms of $\mathcal{T}_d$: bijections of the vertices that preserve incidence. The set Aut $\mathcal{T}_d$ of all automorphisms of $\mathcal{T}_d$ is a group with respect to composition between functions.

The stabilizer of $\mathrm{Aut}\,\mathcal{T}_d$



- The nth level stabilizer $\mathrm{st}(n)$ fixes all vertices up to level n.

The stabilizer of Aut $\mathcal{T}_d$



- The nth level stabilizer $\mathrm{st}(n)$ fixes all vertices up to level n.
- If $H \leq \mathrm{Aut}\,\mathcal{T}$, we define $\mathrm{st}_H(n) = H \cap \mathrm{st}(n)$.

Elements of Aut $\mathcal{T}_d$

- Note $\operatorname{Aut}\mathcal{T}_d = \mathrm{st}(1) \rtimes S_d$.

Elements of Aut $\mathcal{T}_d$

- Note Aut $\mathcal{T}_d = \mathrm{st}(1) \rtimes S_d$.
- We consider the isomorphism $\psi : \mathrm{st}(1) \longrightarrow \mathrm{Aut}\,\mathcal{T}_d \times \overset{d}{\cdots} \times \mathrm{Aut}\,\mathcal{T}_d$.

Elements of $\operatorname{Aut}\mathcal{T}_d$

- Note $\operatorname{Aut}\mathcal{T}_d = \mathrm{st}(1) \rtimes S_d$.
- We consider the isomorphism $\psi : \mathrm{st}(1) \longrightarrow \operatorname{Aut}\mathcal{T}_d \times \overset{d}{\cdots} \times \operatorname{Aut}\mathcal{T}_d$.

Then

$$\begin{aligned}\operatorname{Aut}\mathcal{T}_d = \mathrm{st}(1) \rtimes S_d &\cong (\operatorname{Aut}\mathcal{T}_d \times \overset{d}{\cdots} \times \operatorname{Aut}\mathcal{T}_d) \rtimes S_d \\ &\cong \operatorname{Aut}\mathcal{T}_d \wr S_d \cong ((\cdots \wr S_d) \wr S_d) \wr S_d.\end{aligned}$$

Elements of $\operatorname{Aut} \mathcal{T}_d$

- Note $\operatorname{Aut} \mathcal{T}_d = \operatorname{st}(1) \rtimes S_d$.
- We consider the isomorphism $\psi : \operatorname{st}(1) \longrightarrow \operatorname{Aut} \mathcal{T}_d \times \overset{d}{\cdots} \times \operatorname{Aut} \mathcal{T}_d$.

Then

$$\begin{aligned} \operatorname{Aut} \mathcal{T}_d = \operatorname{st}(1) \rtimes S_d &\cong (\operatorname{Aut} \mathcal{T}_d \times \overset{d}{\cdots} \times \operatorname{Aut} \mathcal{T}_d) \rtimes S_d \\ &\cong \operatorname{Aut} \mathcal{T}_d \wr S_d \cong ((\cdots \wr S_d) \wr S_d) \wr S_d. \end{aligned}$$

- Examples.

A famous example: the Grigorchuk group

A famous example: the Grigorchuk group

$$\Gamma = \langle a, b, c, d \rangle$$

A famous example: the Grigorchuk group

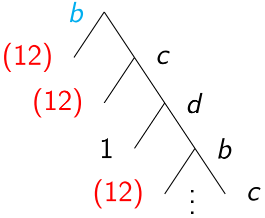
$$\Gamma = \langle a, b, c, d \rangle$$

$$a = (12),\ b = (a, c)\varepsilon,\ c = (a, d)\varepsilon,\ d = (1, b)\varepsilon$$

A famous example: the Grigorchuk group

$$\Gamma = \langle a, b, c, d \rangle$$

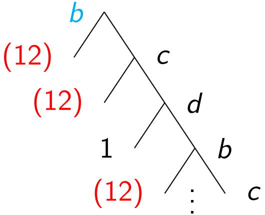
$$a = (12),\ b = (a, c)\varepsilon,\ c = (a, d)\varepsilon,\ d = (1, b)\varepsilon$$

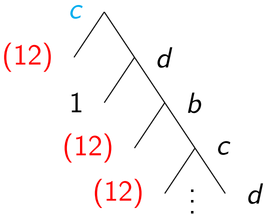


A famous example: the Grigorchuk group

$$\Gamma = \langle a, b, c, d \rangle$$

$$a = (12),\ b = (a, c)\varepsilon,\ c = (a, d)\varepsilon,\ d = (1, b)\varepsilon$$

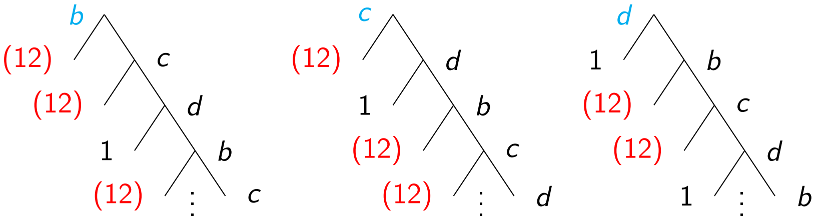




A famous example: the Grigorchuk group

$$\Gamma = \langle a, b, c, d \rangle$$

$$a = (12),\ b = (a, c)\varepsilon,\ c = (a, d)\varepsilon,\ d = (1, b)\varepsilon$$



What is known about Engel elements in subgroups of $\operatorname{Aut}\mathcal{T}$: the case of the Grigorchuk group Γ

What is known about Engel elements in subgroups of $\mathrm{Aut}\,\mathcal{T}$: the case of the Grigorchuk group Γ

- Let $H = \Gamma \wr D_8$. Then $L(H)$ is not a subgroup. (Bludov, 2006)

What is known about Engel elements in subgroups of $\operatorname{Aut}\mathcal{T}$: the case of the Grigorchuk group Γ

- Let $H = \Gamma \wr D_8$. Then $L(H)$ is not a subgroup. (Bludov, 2006)
- $L(\Gamma) = \{x \in \Gamma \mid x^2 = 1\}$. In particular, Γ is not an Engel group. (Bartholdi, 2016)

What is known about Engel elements in subgroups of $\operatorname{Aut}\mathcal{T}$: the case of the Grigorchuk group Γ

- Let $H = \Gamma \wr D_8$. Then $L(H)$ is not a subgroup. (Bludov, 2006)
- $L(\Gamma) = \{x \in \Gamma \mid x^2 = 1\}$. In particular, Γ is not an Engel group. (Bartholdi, 2016)

Moreover:

Theorem (–, Tortora, 2018)

$$\bar{L}(\Gamma) = R(\Gamma) = \bar{R}(\Gamma) = 1$$

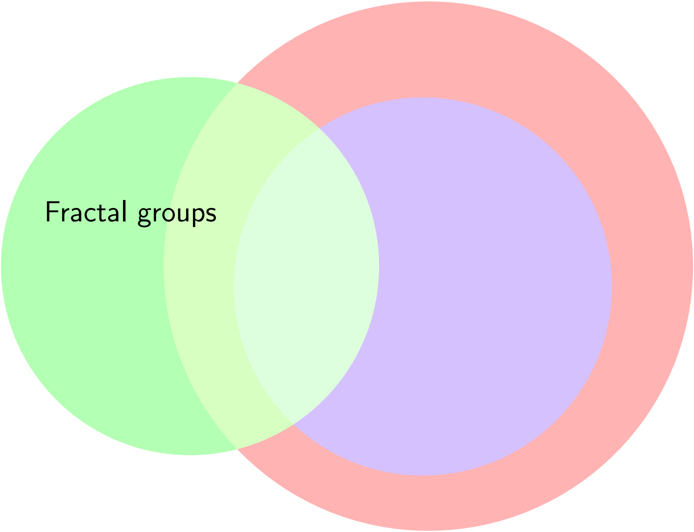
What I want to present today

What I want to present today

$\mathrm{Aut}\,\mathcal{T}$

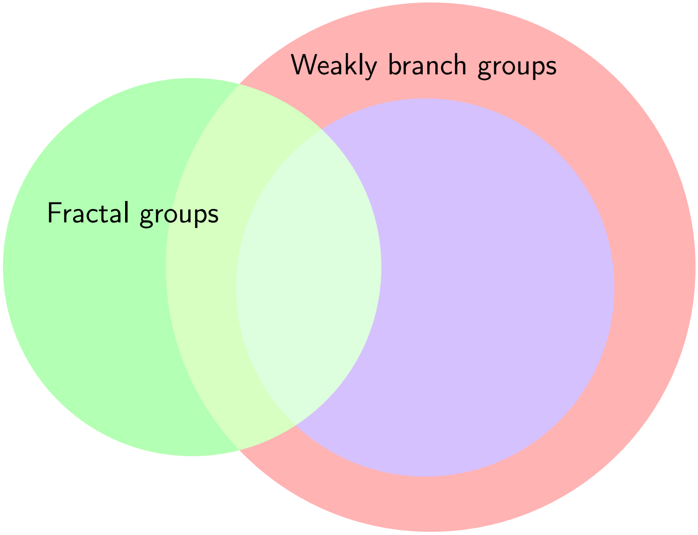
What I want to present today

Aut $\mathcal{T}$



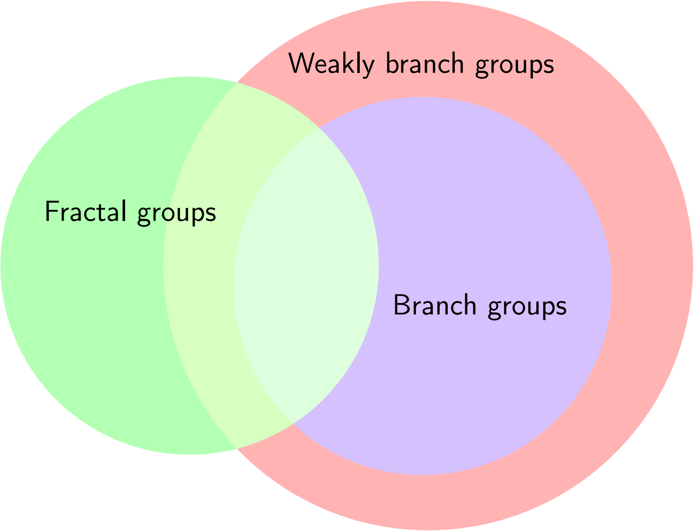
What I want to present today

$\operatorname{Aut}\mathcal{T}$



What I want to present today

Aut $\mathcal{T}$



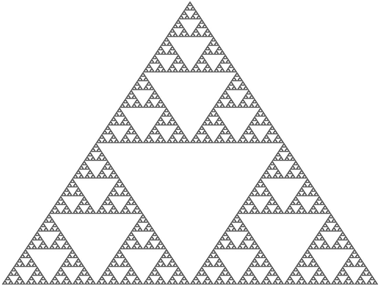
Outline

Fractal groups

Fractal groups

Fractal groups!?

Fractal groups!?



$L(G)$ in fractal groups

Proposition (Fernández-Alcober, Garreta, —, 2018)

Let $G \leq \operatorname{Aut}\mathcal{T}$ *be a fractal group such that* $L(G) \subseteq \operatorname{st}_G(1)$*. Then* $L(G) = 1$*.*

$L(G)$ in fractal groups

Proposition (Fernández-Alcober, Garreta, —, 2018)

Let $G \leq \operatorname{Aut} \mathcal{T}$ be a fractal group such that $L(G) \subseteq \mathrm{st}_G(1)$. Then $L(G) = 1$.

Idea of the proof:

- Step 1: $L(G) \cap \mathrm{st}_G(1) = \{h \in \mathrm{st}_G(1) \mid \psi(h) \in L(G) \times \overset{d}{\cdots} \times L(G)\}$.
- Step 2: if $S \subseteq \mathrm{st}(1)$ and $\psi(S) \subseteq S \times \overset{d}{\cdots} \times S$, then $S = 1$.

The set $L(G)$

Theorem (Fernández-Alcober, Garreta, —, 2018)

Let G be a fractal group such that $|G' : \mathrm{st}_G(1)'| = \infty$. Then $L(G) = 1$.

The set $L(G)$

Theorem (Fernández-Alcober, Garreta, —, 2018)

Let G be a fractal group such that $|G' : \mathrm{st}_G(1)'| = \infty$. Then $L(G) = 1$.

Moreover:

Theorem (Fernández-Alcober, Garreta, —, 2018)

Let G be a fractal group with torsion-free abelianization. Then $L(G) = 1$.

Application to specific fractal groups

Some examples:

- The Basilica group;
- The Brunner Sidki Vieira group;
- The GGS group $\mathcal{G}$ with constant defining vector.

Outline

Branch groups

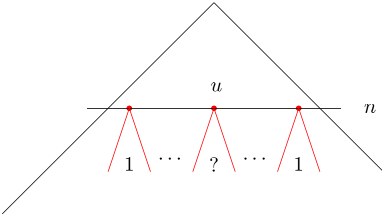
Given $G \leq \operatorname{Aut}\mathcal{T}$, can we find for every $n \in \mathbb{N}$ a subgroup of $\mathrm{st}_G(n)$ which is a direct product?

Branch groups

Given $G \leq \operatorname{Aut} \mathcal{T}$, can we find for every $n \in \mathbb{N}$ a subgroup of $\mathrm{st}_G(n)$ which is a direct product?

The *rigid stabilizer* of the vertex u is

$$\mathrm{rst}_G(u) = \{g \in G : g \text{ fixes all vertices outside } \mathcal{T}_u\}$$

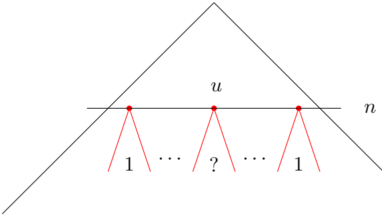


Branch groups

Given $G \leq \operatorname{Aut}\mathcal{T}$, can we find for every $n \in \mathbb{N}$ a subgroup of $\mathrm{st}_G(n)$ which is a direct product?

The *rigid stabilizer* of the vertex u is

$$\mathrm{rst}_G(u) = \{g \in G : g \text{ fixes all vertices outside } \mathcal{T}_u\}$$



The *rigid stabilizer* of the nth level is

$$\mathrm{rst}_G(n) = \prod_{u \in X^n} \mathrm{rst}_G(u).$$

Definition of branch groups

- We say that G is a *branch group* if for all $n \geq 1$, the index of the rigid nth level stabilizer in G is finite.

Definition of branch groups

- We say that G is a *branch group* if for all $n \geq 1$, the index of the rigid nth level stabilizer in G is finite.
- We say that G is a *weakly branch group* if all of its rigid vertex stabilizers are nontrivial for every vertex of the tree.

Definition of branch groups

- We say that G is a *branch group* if for all $n \geq 1$, the index of the rigid nth level stabilizer in G is finite.
- We say that G is a *weakly branch group* if all of its rigid vertex stabilizers are nontrivial for every vertex of the tree.
- Branch $\implies$ Weakly branch.

$L(G)$ in branch groups

$L(G)$ in branch groups

Theorem (Fernández-Alcober, —, Tracey, 2019)

Let G be a branch group. Then

Theorem (Fernández-Alcober, —, Tracey, 2019)

Let G be a branch group. Then

- $\bar{L}(G) = 1$.

$L(G)$ in branch groups

Theorem (Fernández-Alcober, —, Tracey, 2019)

Let G be a branch group. Then

- $\bar{L}(G) = 1$.
- *If G contains an element of infinite order, then $L(G) = 1$.*

$L(G)$ in branch groups

Theorem (Fernández-Alcober, —, Tracey, 2019)

Let G be a branch group. Then

- $\bar{L}(G) = 1$.
- *If G contains an element of infinite order, then $L(G) = 1$.*
- *If G is periodic, then $L(G)$ is a p-set for some prime $p = p(G)$.*

$L(G)$ in branch groups

Theorem (Fernández-Alcober, —, Tracey, 2019)

Let G be a branch group. Then

- $\bar{L}(G) = 1$.
- *If G contains an element of infinite order, then $L(G) = 1$.*
- *If G is periodic, then $L(G)$ is a p-set for some prime $p = p(G)$.*
- *If $L(G) \neq 1$, then G is virtually a p-group.*

Some applications

- Aut $\mathcal{T}_d$;

Some applications

- Aut $\mathcal{T}_d$;
- Finitary automorphisms;
- Multi-edge spinal groups;
- Hanoi towers group.

$R(G)$ in weakly branch groups

Theorem (Fernández-Alcober, —, Tracey, 2019)

Let G be a weakly branch group acting on $\mathcal{T}$. If $\mathrm{rst}_G(n)$ is not Engel for any n, then $R(G) = 1$.

$R(G)$ in weakly branch groups

Theorem (Fernández-Alcober, —, Tracey, 2019)

Let G be a weakly branch group acting on $\mathcal{T}$. If $\mathrm{rst}_G(n)$ is not Engel for any n, then $R(G) = 1$.

Moreover:

Conjecture

Let G be a finitely generated weakly branch group. Then $R(G) = 1$.

Outline

Some remarks

$L(G) = 1$ completes the description of Engel elements in G:

- $R(G) = 1$;
- $\bar{L}(G) = 1$.

(Because $R^{-1}(G), \bar{L}(G) \subseteq L(G)$)

Some remarks

$L(G) = 1$ completes the description of Engel elements in G:

- $R(G) = 1$;
- $\bar{L}(G) = 1$.

(Because $R^{-1}(G), \bar{L}(G) \subseteq L(G)$)

Non solvable fractal/weakly branch groups have few Engel elements.

Some remarks

$L(G) = 1$ completes the description of Engel elements in G:

- $R(G) = 1$;
- $\bar{L}(G) = 1$.

(Because $R^{-1}(G), \bar{L}(G) \subseteq L(G)$)

Non solvable fractal/weakly branch groups have few Engel elements.

Golod-Shafarevich groups cannot be branch.

Further work

An open question:

Further work

An open question:

Can a finitely generated branch group be Engel?

Further work

An open question:

Can a finitely generated branch group be Engel?

Some possible approaches:

Further work

An open question:

Can a finitely generated branch group be Engel?

Some possible approaches:

- Determine this "p-set" of left Engel elements.
- Reduce to the periodic case and prove that rigid stabilizers are not Engel.
- Prove that branch Lie algebras are not Engel.

Grazie!

Eskerrik asko :)

CS ;)