A graph approach to the structure of locally inverse semigroups given by presentations

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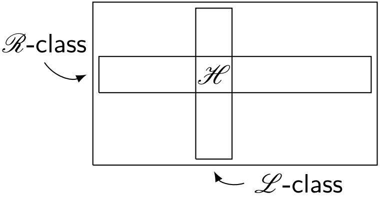
- $E(S)$ - set of idempotents of S.
- $V(x) = \{x' \in S \mid xx'x = x \text{ and } x'xx' = x'\}$ - set of inverses of $x \in S$.

- Group: $|E(S)| = 1$ and $|V(x)| = 1$ for all $x \in S$.
- Inverse semigroup: $|V(x)| = 1$ for all $x \in S$.
- Regular semigroup: $V(x) \neq \emptyset$ for all $x \in S$.
- Locally inverse sem. = regular + eSe inverse subsem. for all $e \in E(S)$.

- Green's relations on S regular:
  - $a \mathscr{R} b \quad \Leftrightarrow \quad aS = bS$.
  - $a \mathscr{L} b \quad \Leftrightarrow \quad Sa = Sb$.
  - $a \mathscr{J} b \quad \Leftrightarrow \quad SaS = SbS$.
  - $\mathscr{D} = \mathscr{R} \vee \mathscr{L}$.
  - $\mathscr{H} = \mathscr{R} \cap \mathscr{L}$.

Egg-box representation of a $\mathscr{D}$-class



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Theorem (Yeh'92)

An e-variety has bifree objects on every set X if and only if it is constituted by locally inverse semigroups or by regular E-solid semigroups only.

$BFLI(X)$ - bifree locally inverse semigroup on X.

- Every loc. inv. sem. is a homomorphic image of some $BFLI(X)$.

- X - non-empty set.
- $X' = \{x' : x \in X\}$ - disjoint copy of X (a set of formal inverses).
- $\overline{X} = X \cup X'$ ($'$ is viewed as an involution on $\overline{X}$: $y' = x$ if $y = x' \in X'$).
- $\widehat{X} = \overline{X} \cup \{(x \wedge y) \mid x, y \in \overline{X}\}$.

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Presentation for locally inverse semigroups: pair $\langle X; R \rangle$ where $R \subseteq \widehat{X}^+ \times \widehat{X}^+$.

- $LI\langle X; R \rangle = \widehat{X}^+/\rho$ where ρ is the congruence on $\widehat{X}^+$ generated by $\theta \cup R$.

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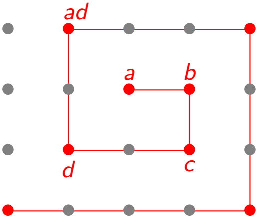
Corollary

*$LI\langle X; R\rangle \in$ **LI** and every $S \in$ **LI** can be described in this manner.*

Example: the 4-spiral semigroup Sp_4

Sp_4 is generated by idempotents $\{a, b, c, d\}$ such that $a \mathscr{R} b \mathscr{L} c \mathscr{R} d = da$.



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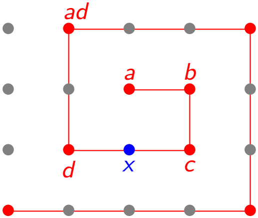

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$$Sp_4 = LI\langle \{x\};\ \{x' = x'^2,\ x = (x \wedge x)x\} \rangle .$$

- $b = x'$ and $d = x \wedge x$.

Group presentations $\longrightarrow$ Cayley graphs.

Inverse monoid presentations $\rightarrow$ Schützenberger graphs (Stephen'90):

- maximal strongly connected components of the Cayley graph;
- each Schützenberger graph records the information about the partial multiplication on the right, inside an $\mathscr{R}$-class R, by elements of a generating set $\overline{X}$;
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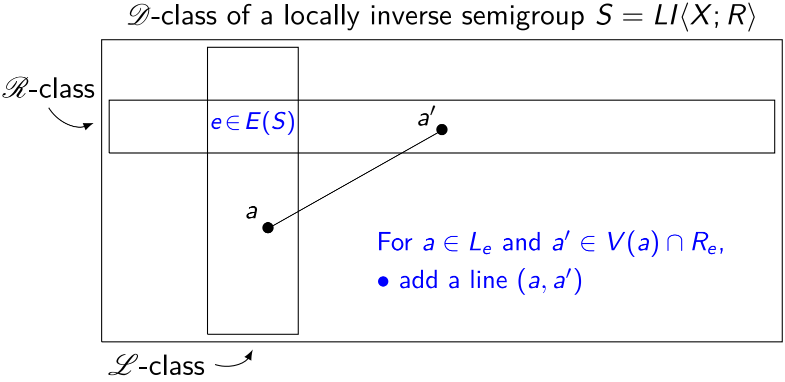
But if we go beyond the class of all inverse semigroups, the information about the partial multiplication on the right is no longer sufficient to characterize the $\mathscr{D}$-classes.

A graph approach: the idea



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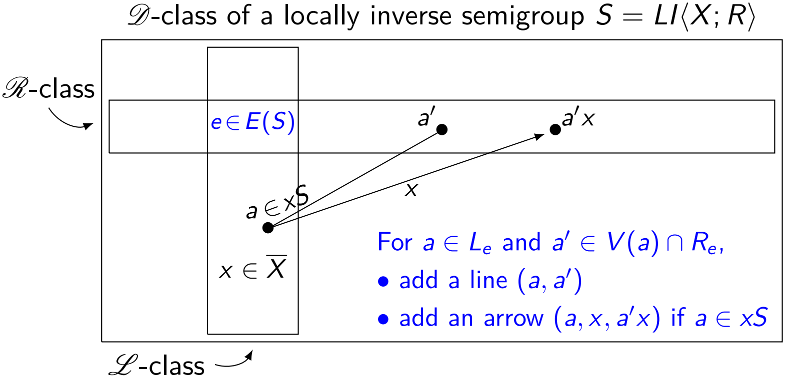

- (S loc. inverse) If $a \in xS$ and $a', a'_1 \in V(a) \cap R_e$, then $a'x = a'_1x$.

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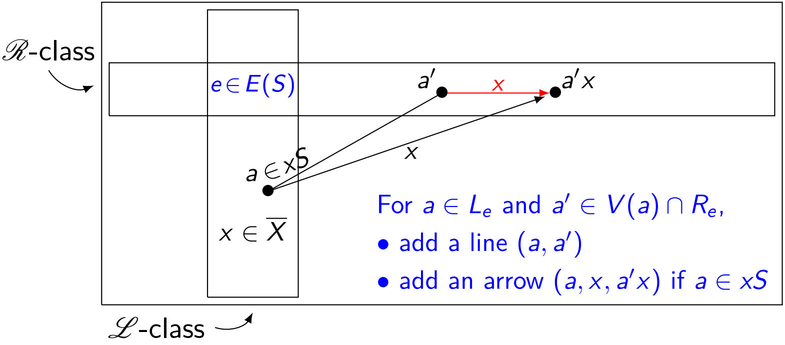
$\mathscr{D}$-class of a locally inverse semigroup $S = LI\langle X; R\rangle$



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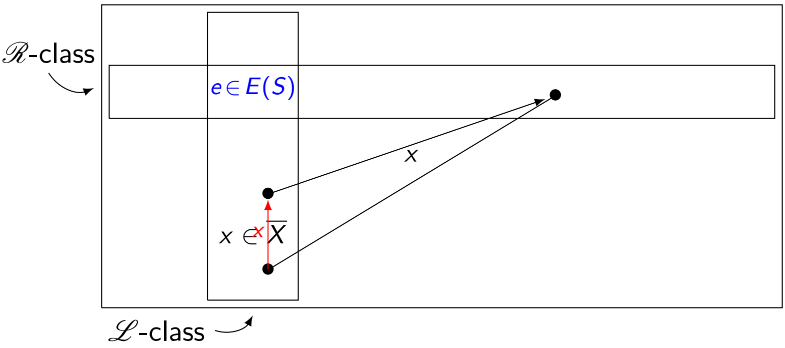

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An oriented bipartite graph is a (simple) graph $\Gamma = (\mathscr{V}, \mathscr{E})$ with vertices $\mathscr{V}$ and edges $\mathscr{E}$ such that:

- $\mathscr{V} = \mathscr{V}_l \,\dot\cup\, \mathscr{V}_r$: $\mathscr{V}_l$ - left vertices; $\mathscr{V}_r$ - right vertices.
- $\mathscr{E} = \overline{\mathscr{E}} \,\dot\cup\, \vec{\mathscr{E}}$.
- $\overline{\mathscr{E}} \subseteq \mathscr{V}_l \times \mathscr{V}_r$: set of lines which are represented by pairs $(\mathfrak{a}, \mathfrak{b})$.
- $\vec{\mathscr{E}} \subseteq \mathscr{V}_l \times \overline{X} \times \mathscr{V}_r$: set of arrows which are represented as triples $(\mathfrak{a}, x, \mathfrak{b})$
  - each arrow is oriented from a vertex $\mathfrak{a} \in \mathscr{V}_l$ to a vertex $\mathfrak{b} \in \mathscr{V}_r$ and labeled with a letter x from $\overline{X}$.

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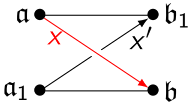
The content of a vertex $\mathfrak{a} \in \mathscr{V}$ is the set $\mathrm{c}(\mathfrak{a})$ of all letters labeling arrows of Γ with $\mathfrak{a}$ as one of its endpoints.

A locally inverse word graph (liw-graph) is a (strongly) connected oriented bipartite graph such that

- $\mathrm{c}(\mathfrak{a}) \neq \emptyset$;
- if $(\mathfrak{a}, x, \mathfrak{b}) \in \vec{\mathscr{E}}$, then there are lines $(\mathfrak{a}, \mathfrak{b}_1)$ and $(\mathfrak{a}_1, \mathfrak{b})$ in $\overline{\mathscr{E}}$ such that $(\mathfrak{a}_1, x', \mathfrak{b}_1) \in \vec{\mathscr{E}}$.



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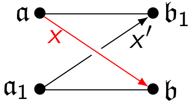
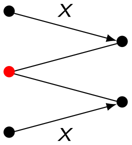
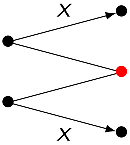


An liw-graph is called reduced if it avoids the following configurations:





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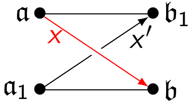

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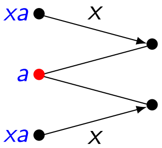

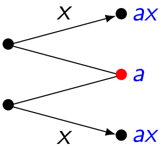

The locally inverse word graphs Γ_e

For $S = LI\langle X; R\rangle$ and $e \in E(S)$, define $\Gamma_e = (\mathscr{V}, \mathscr{E})$ as follows:

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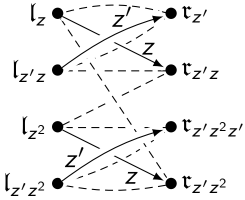
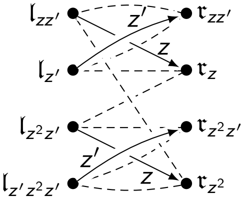
Proposition (LO)

Γ_e is a reduced liw-graph.

Example: $S = LI\langle \{z\};\ \{z = z^3,\ z' = z'^2\}\rangle$.



$\Gamma_{z'z}$



$\Gamma_{zz'}$

$z'z^2$ $z'z\,*$	$z'z^2z'$ $z'\,*$
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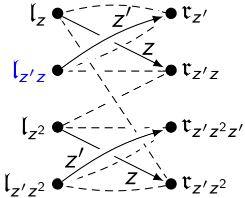

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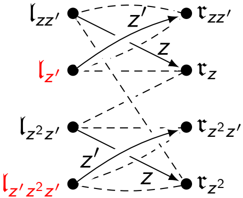

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$z'z^2$ $z'z$ *	$z'z^2z'$ z' *
z^2 * z	z^2z' zz' *

Theorem (LO)

1. *For $e, f \in E(S)$, $e \mathscr{D} f$ if and only if Γ_e and Γ_f are isomorphic.*
2. *For each $a \in R_e \cap L_f$, there exists a unique isomorphism $\varphi : \Gamma_e \to \Gamma_f$ such that $\mathfrak{l}_e\varphi = \mathfrak{l}_a$; and there are no other isomorphisms from Γ_e to Γ_f.*

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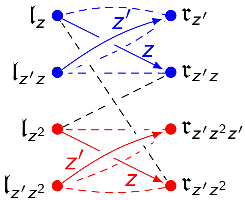

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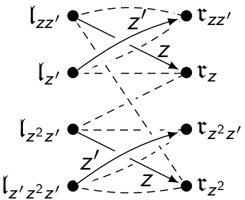

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3. *For each $e \in E(S)$, the maximal subgroups of D_e are isomorphic to $Aut\,\Gamma_e$.*

The birooted locally inverse word graphs $\mathfrak{A}_s$

A birooted liw-graph is a triple $\mathfrak{A} = (\mathfrak{l}, \Gamma, \mathfrak{r})$ where Γ is an liw-graph and $(\mathfrak{l}, \mathfrak{r}) \in \mathscr{V}_l \times \mathscr{V}_r$.

- $\mathfrak{l}$ is the left root of $\mathfrak{A}$ and we denote it by $\mathfrak{l}(\mathfrak{A})$.
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Let $e, f \in E(S)$ and $a, a_1, b, b_1 \in S$ such that $a \mathscr{L} e \mathscr{R} b$ and $a_1 \mathscr{L} f \mathscr{R} b_1$. Then $(\mathfrak{l}_a, \Gamma_e, \mathfrak{r}_b)$ is isomorphic (as birooted liw-graphs) to $(\mathfrak{l}_{a_1}, \Gamma_f, \mathfrak{r}_{b_1})$ if and only if $e \mathscr{D} f$ and $ab = a_1 b_1$.

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Thus, for each $s \in S = LI\langle X; R\rangle$, we define (up to isomorphism)

$$\mathfrak{A}_s = (\mathfrak{l}_a, \Gamma_e, \mathfrak{r}_b)$$

where $e \in E(S) \cap D_s$, $a \mathscr{L} e \mathscr{R} b$ and $ab = s$.

Proposition (LO)

Let $s, t \in S = LI\langle X; R\rangle$. Then:

- $s = t \quad \Leftrightarrow \quad \mathfrak{A}_s$ *and* $\mathfrak{A}_t$ *are isomorphic.*
- $s \mathscr{R} t \quad \Leftrightarrow \quad \mathfrak{A}_s$ *and* $\mathfrak{A}_t$ *are "left" isomorphic.*
- $s \mathscr{L} t \quad \Leftrightarrow \quad \mathfrak{A}_s$ *and* $\mathfrak{A}_t$ *are "right" isomorphic.*
- $s \mathscr{D} t \quad \Leftrightarrow \quad \mathfrak{A}_s$ *and* $\mathfrak{A}_t$ *are "weakly" isomorphic.*

- $s \mathscr{H} t \quad \Leftrightarrow \quad \mathfrak{A}_s$ *and* $\mathfrak{A}_t$ *are left and right isomorphic.*
- $s \mathscr{J} t \quad \Leftrightarrow \quad$ *there exist weak hom.* $\varphi : \mathfrak{A}_s \to \mathfrak{A}_t$ *and* $\psi : \mathfrak{A}_t \to \mathfrak{A}_s$.

- $s \leq t \quad \Leftrightarrow \quad$ *there exist a homomorphism* $\varphi : \mathfrak{A}_t \to \mathfrak{A}_s$.

Proposition (LO)

- $s \in E(S) \quad \Leftrightarrow \quad (\mathfrak{l}(\mathfrak{A}_s), \mathfrak{r}(\mathfrak{A}_s)) \in \overline{\mathscr{E}}(\mathfrak{A}_s).$
- $s \in \langle E(S) \rangle \quad \Leftrightarrow \quad \mathfrak{l}(\mathfrak{A}_s)$ *and* $\mathfrak{r}(\mathfrak{A}_s)$ *are 'line-connected' in* $\mathfrak{A}_s$.

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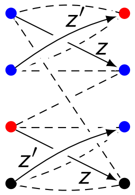
- $t \in V(s) \quad \Leftrightarrow \quad$ *there exist a weak isomorphism* $\varphi : \mathfrak{A}_s \to \mathfrak{A}_t$ *such that* $(\mathfrak{l}\varphi, \mathfrak{r}_1)$ *and* $(\mathfrak{l}_1, \mathfrak{r}\varphi)$ *are lines in* $\mathfrak{A}_t$.

Returning to S:



$\mathfrak{A}_{z^2 z'}$

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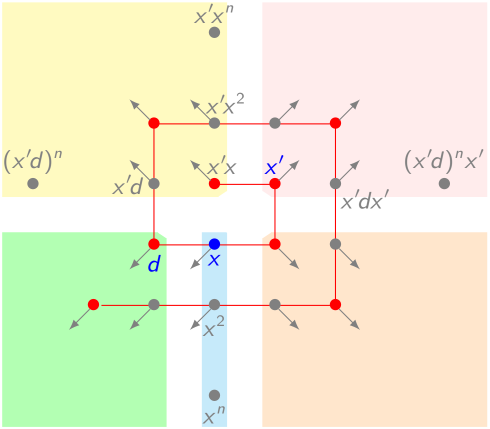
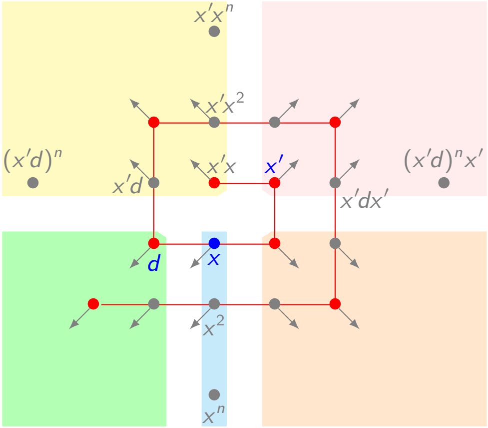
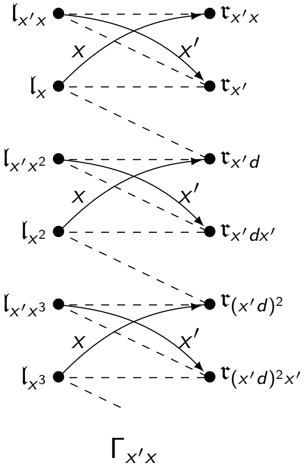


Example: returning to the 4-spiral semigroup Sp_4

$Sp_4 = LI\langle \{x\};\ \{x' = x'^2, x = (x \wedge x)x\} \rangle \qquad (d = x \wedge x)$





L. Oliveira, Presentations for locally inverse semigroups, arXiv:1903.10236 [math.GR]

Thank you for your attention