Factorization Theorems

for Finite Groups of Characteristic p

SandGAL 2019
Cremona
June 10 – 13, 2019

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Introduction

Definition: A finite group G is of characteristic p if $C_G(O_p(G)) \leq O_p(G)$.

Glauberman's ZJ-Theorem (1968): Let G be a finite group of characteristic p and $S \in Syl_p(G)$. Suppose $p \neq 2$ and G is p-stable. Then $Z(J(S)) \trianglelefteq G$

If the stability condition is dropped

Basic counterexamples: $q^2SL_2(q)$

[Target: study the "counterexamples"]

Notation

G is a finite group, p a prime divisor of $|G|$ and $S \in Syl_p(G)$.

$$\mathbb{Z}(\mathbf{T}) := \Omega_1(Z(T)) \text{ for all } T \leq S$$

$$\mathbf{C} := C_G(\mathbb{Z}(S))$$

$$\mathbf{B_S} := C_{O_p(C)}(\mathbb{Z}(J(O_p(C))))$$

$$\mathbf{Bau_p(G)} := \{B_S^g \mid g \in G\}$$

Baumann component: Let G be a finite group of characteristic p and $B \in Bau_p(G)$. Then E is a Baumann component of G if E is minimal subject to

$$E \trianglelefteq\trianglelefteq G \text{ and } E \not\leq N_G(B).$$

$\mathcal{E}\ (G) :=$ the set of the Baumann components of G

Plan: Let B be a finite p-group and let W be a non-trivial characteristic subgroup of B.

The best would be:

$(*)$ $W \trianglelefteq G$ for all finite groups G of characteristic p with $B \in Bau_p(G)$.

This is impossible.

Best hope: describe all counterxamples to $(*)$.

More precisely, give the isomorphism types of all Baumann components of G not normalizing W.

And that is what we intend to do and what I mean by a counterexample.

In this talk I will describe general factorization theorems which allow to narrow down the structure of such counterexamples, and where one does not need to specify which characteristic subgroup W one actually want to use.

Theorem A:

Let G be a finite $\mathcal{CK}$-group of characteristic p and $B \in Bau_p(G)$.
Then

(a) $G = \langle \mathcal{E}(G) \rangle N_G(B) = \langle Bau_p(G) \rangle N_G(B)$

(b) $B \trianglelefteq G \iff \mathcal{E}(G) = \emptyset$

Hence if $W \neq 1$ is characteristic in B then

(a) $\Rightarrow$ [$W \trianglelefteq G \iff W \trianglelefteq \langle Bau_p(G) \rangle$]

(b) $\Rightarrow$ [$\mathcal{E}\ (G) = \emptyset \Rightarrow W \trianglelefteq G$]

In a "counterexample" we may assume that

$$G = \langle Bau_p(G) \rangle \text{ and } \mathcal{E}(G) \neq \emptyset$$

$\mathcal{C} := \mathcal{C}(B) :=$ the class of finite $\mathcal{CK}$-groups H of characteristic p with $B \in Bau_p(H)$ such that $H = \langle Bau_p(H) \rangle$ and $|\mathcal{E}(H)| = 1$

Theorem B:

Let G be a finite group of characteristic p with $\mathcal{E}(G) \neq \emptyset$.
Let $B \in Bau_p(G)$ and $E \in \mathcal{E}(G)$.
Put $F := \langle E, B \rangle$ and $\widehat{G} := G/O_p(G)$.

Then $E \trianglelefteq F$ and

(a) $[\widehat{E}, \widehat{L}] = 1$ *and* $[\mathbb{Z}(O_p(G)), E, L] = 1$ *for all* $L \in \mathcal{E}(G)$ *with* $L \neq E$.
(b) *If G is a $\mathcal{CK}$-group then $F \in \mathcal{C}$.*

Hence the isomorphism type of Baumann components can be studied in groups in $\mathcal{C}$.

$\mathcal{D} := \mathcal{D}(B) :=$ class of $H \in \mathcal{C}$ such that for $T \in Syl_p(H)$ with $B := B_T$

$$C_{\widehat{T}}(\widehat{B}) \leq \widehat{B} \text{ where } \widehat{H} := G/O_p(H)$$

Theorem 1 (Baumann argument): *Let B be a finite p-group and $G \in \mathcal{D}$. Then $O_p(G) \leq B$.*

[Corollary. $W := \langle \mathbb{Z}(O_p(G_i)) \mid G_i \in \mathcal{D} \rangle \leq B$]

For a finite group G, set

$$\mathcal{D}_G := \mathcal{D}_G(B) := \{H \in \mathcal{D}(B) \mid H \leq G\}$$

Theorem 2:

Let B be a finite p-group and $G \in \mathcal{C}$. Put $T := BO_p(G)$,

$$\mathbb{V} := [\mathbb{Z}(O_p(G)), G],\ \widetilde{\mathbb{V}} := \mathbb{V}/C_{\mathbb{V}}(G) \text{ and } \overline{G} := G/O_p(G)$$

Then $C_G(\mathbb{V}) = O_p(G)$ and $N_G(T) = N_G(B)$.

Moreover one of the following holds, where q is a power of p:

Case (1) $\mathcal{D}_G = \{G\}$ *and*

(a) $\overline{G} \cong SL_n(q)$, $n \geq 2$, $Sp_{2n}(q)$, $n \geq 1$ *or* $G_2(q)$ *and* $p = 2$

(b) $\widetilde{\mathbb{V}}$ *is a natural module for* $\overline{G}$

Case (2) $G \notin \mathcal{D}$ *and*

(a) (i) $\overline{G} \cong Sp_{2n}(q)$, $n \geq 2$, *or*

(ii) $\overline{G} \cong Sym(n)$, $p = q = 2$ *and* $n \geq 7$, $n \equiv 2, 3 \pmod 4$

(b) $G = \langle \mathcal{D}_G \rangle O_p(G)$ *and* $N_G(B)$ *acts transitively on* $\mathcal{D}_G$

(c) *For all* $L \in \mathcal{D}_G$

(i) $\overline{L} \cong SL_2(q)$

(ii) $[\widetilde{\mathbb{V}}, L]$ *is a natural module for* $\overline{L}$

(iii) $\widetilde{\mathbb{V}} = [\widetilde{\mathbb{V}}, L] \times \widetilde{C_{\mathbb{V}}(L)}$ *for all* $L \in \mathcal{D}_G$

Corollary 3:

Let B be a finite p-group and G a finite $\mathcal{CK}$-group of characteristic p with $B \in Bau_p(G)$. Then

$$G = N_G(B)\langle\mathcal{D}_G\rangle.$$

Variations of Thompson and Baumann subgroups

Let T be a finite p-group and $A, B \leq T$.

A is an **offender on** B if $|B||C_A(B)| \leq |A||C_B(A)|$.

Put

$\mathcal{L}(T) :=$ the set of $A \leq T$ s.t. A is an offender on each $B \leq T$

$\mathcal{O}(T) :=$ the set of the elementary abelian subgroups in $\mathcal{L}(T)$ of maximal order

$$J^{\diamond}(T) := \langle \mathcal{O}(T) \rangle \quad \text{and} \quad B^{\diamond}(T) := C_T(C_T(J^{\diamond}(T)))$$

Theorem 4: *Let B be a finite p-group and $G \in \mathcal{C}$. Put*

$$B^\diamond := B^\diamond(BO_p(G)) \text{ and } G^\diamond := \langle B^{\diamond G} \rangle.$$

Then one of the following holds:

(a) $\mathcal{D}_G(B^\diamond) = \emptyset$ *and* $B^\diamond \trianglelefteq G$.

(b) $\mathcal{D}_G(B^\diamond) = \{G^\diamond\}$, $\mathcal{E}(G^\diamond) = \{E\}$ *and* $G^\diamond = B^\diamond E$.

(c) $G^\diamond \notin \mathcal{D}(B^\diamond)$ *and*

(i) $\overline{G} \cong Sp_{2n}(q)$, $n \geq 2$, *or* $Sym(n)$, $n \geq 7$,

(ii) $\overline{L} \cong SL_2(q)$ *for* $L \in \mathcal{D}_G(B^\diamond)$.

(iii) $N_G(B) = N_G(B^\diamond)$.

Corollary 5:

Let B be a finite p-group and G a finite $\mathcal{CK}$-group of characteristic p with $B \in Bau_p(G)$. Put $B^\diamond := B^\diamond(B)$. Then

$$G = N_G(\, B^\diamond\,)\langle\, \mathcal{D}_G(B^\diamond)\,\rangle.$$