On totally synchronizing digraphs

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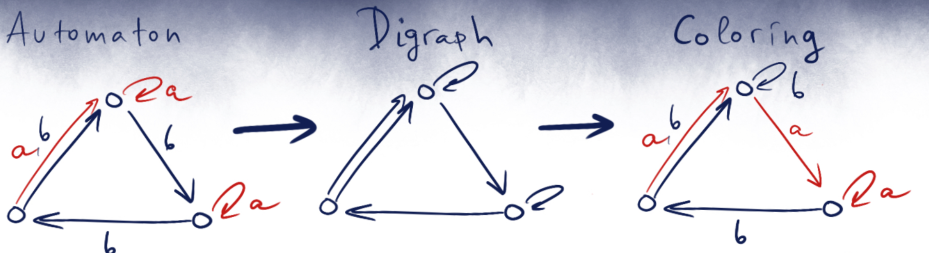
Automata vs Digraphs



Automaton = colored digraph. $p.a=q$ iff $p \xrightarrow{a} q$

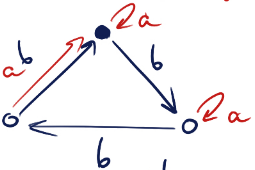
Digraphs are k-out regular, k is the number of input letters

Digraph → coloring with specific properties?

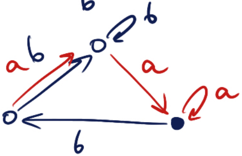
Synchronization

A word w is synchronizing for the automaton $\mathcal{A}$, if it labels a path from each state to a particular one.

An automaton is synchronizing, if it possesses a synchronizing word

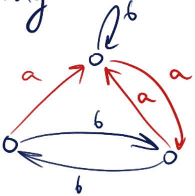


abba is synchronizing



aaa is synchronizing

not synchronizing :



Synchronization: applications

- Restoring control over a system
- Self-synchronizing codes
- Related to primitivity in matrices
- synchronizing groups

A permutation group is synchronizing, if after adding any non-group element, the generated submonoid contains a constant map (transformation of rank 1).

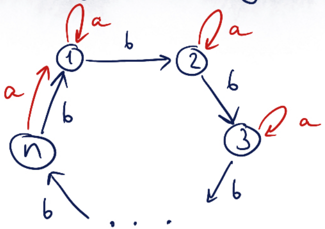
Every 2-transitive group is synchronizing; every synchronizing group is primitive.

(Araújo, Cameron, Steinberg, 2015)

The Černý Conjecture

The Černý conjecture (1964) Every n-state synchronizing automaton possesses a synchronizing word of length at most $(n-1)^2$

Example



synchronizing word:

$$\left(ab^{n-1}\right)^{n-2}a$$

Upper bounds:

$$\frac{n^3-n}{6} \quad \text{(Pin, Frankl, 1983)}$$

$$\frac{85059n^3+90024n^2+196504n-10648}{511104} \quad \text{(Szykuła, 2018)}$$

The Road Coloring Theorem

What structural properties of a digraph can ensure the existence of a synchronizing coloring?

Primitive digraph = strongly connected and gcd of its cycle lengths is 1

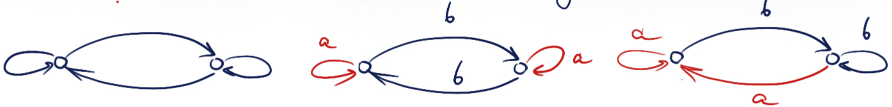
Theorem (Trahtman, 2007) A strongly connected digraph possesses a synchronizing coloring iff it is primitive

Theorem (Béal, Perrin, 2008) There is an $O(kn^2)$-time algorithm to find a synchronizing coloring, where n is the number of vertices, k is the out-degree

Synchronizing ratio

$$\text{synchronizing ratio} = \frac{\#\ \text{synch. colorings}}{\#\ \text{all colorings}}$$

NOTE! The total number of colorings is $(k!)^n$

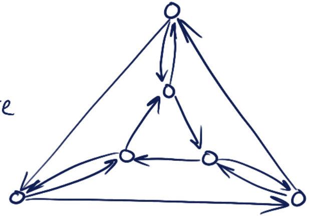


This digraph has 4 colorings, 2 are synchronizing

$\Rightarrow$ synchronizing ratio $= 1/2$

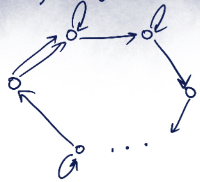
<u>Conjecture</u> (Gusev, Szykula, 2015)

The synchronizing ratio of a primitive k-out regular digraph is at least $\frac{k-1}{k}$ with a single exception having synch. ratio $15/32$



Totally synchronizing digraphs

totally synchronizing = synchronizing ratio equal to 1



Černý digraphs are totally synchronizing

Structural characterization of totally synchronizing digraphs?

<u>Theorem</u> (Kari, 2003) Every primitive Eulerian digraph has both synchronizing and non-synchronizing coloring

<u>Conjecture</u> (Gusev, Szykuła, 2015) For every $k \geq 2$ the probability that a random k-out regular digraph is totally synchronizing goes to 1 as n goes to infinity

The eigenvector

Theorem (Corollary of the Perron-Frobenius Theorem)
Let $A(G)$ be the adjacency matrix of a primitive k-out regular digraph G. There exists a positive integer eigenvector $\vec{v}$ of $A(G)$ such that

$$\vec{v}A(G) = k\vec{v}.$$

We say that $\vec{v}$ is the eigenvector of G.

NOTE: $\vec{v}$ is proportional to the stationary distribution of the random walk on G when each edge is taken with the uniform probability $1/k$

Partitions of the eigenvector

Let $Q = \{1, 2, \ldots, n\}$ and let $\vec{v}[i]$ be the ith entry of $\vec{v}$

The weight of a subset $S \subseteq G$ is given by

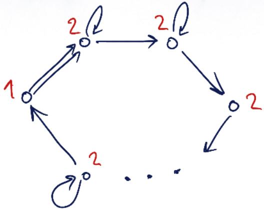
$$wg(S) = \sum_{i \in S} \vec{v}[i]$$

<u>Theorem</u> (Friedman, 1990) Every coloring $\mathcal{A}$ of G has a partition of vertices into synchronizing subsets $Q_1, Q_2, \ldots, Q_\ell$ of maximal weight: $wg(Q_1) = wg(Q_2) = \ldots = wg(Q_\ell)$, and for any synchronizing subset S we have $wg(S) \leq wg(Q_1)$

Non partitionable eigenvectors

Corollary If the eigenvector of a primitive k-out regular digraph G is not partitionable, then G is totally synchronizing.

Example



the Černý digraphs
$\vec{v} = (2, 2, \ldots, 2, 1)$
$\Rightarrow$ totally synchronizing

Theorem (Gusev, P., 2016) If the eigenvector $\vec{v}$ of a digraph G is partitionable, then there exists a non-totally synchronizing digraph G' with the same eigenvector.

Automorphisms

Graph automorphism is a bijection preserving the edges

Automaton automorphism is a graph automorphism preserving the colors of the edges

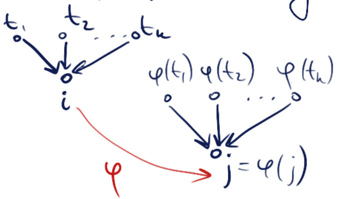
<u>Lemma</u> Let $\mathcal{A}$ be a strongly connected automaton. If $\mathrm{Aut}(\mathcal{A}) \neq 1$, then $\mathcal{A}$ is not synchronizing

<u>Proposition</u> Let G be a strongly connected digraph. If $\mathrm{Aut}(G)$ acts transitively on G, then G is not totally synchronizing

<u>Proof</u> All vertices have the same in-degree

$\Rightarrow$ G is Eulerian

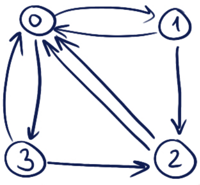
$\Rightarrow$ not totally synchronizing



Automorphisms

Theorem If $\mathrm{Aut}(G)$ of a 2-out regular digraph G is non-trivial and contains an element of order different from a power of 2, then G is not totally synchronizing.

Proposition There is a totally synchronizing digraph G with $\mathrm{Aut}(G) \cong \mathbb{Z}_2$



$\mathrm{Aut}(G) = \{(13), \varepsilon\} \cong \mathbb{Z}_2$

$\vec{v}(G) = (2, 1, 1, 1)$

non-partitionable

$\Rightarrow G$ is totally synchronizing

Thank you!