SandGAL19
Group Theory Session

Generalized Right-Angled Artin pro-p-groups

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Graphs and RAAGs

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ a finite undirected graph with vertices and edges

$$\mathcal{V} = \{v_1, \ldots, v_d\}, \quad \mathcal{E} = \{(v_i, v_j), 1 \leq i < j \leq d\}$$

(thus, no loops). The right-angled Artin group G_Γ associated to the graph Γ is the group

$$G_\Gamma = \langle v_1, \ldots, v_d \mid [v_i, v_j] = 1 \text{ for } (v_i, v_j) \in \mathcal{E} \rangle.$$

They have nice properties!

- G_Γ is torsion-free and $G_\Gamma^{ab} \simeq \mathbb{Z}^d$
- If $\Gamma' \subseteq \Gamma$ is an induced subgraph then $G_{\Gamma'} \leq G_\Gamma$, and if $\Gamma = \Gamma_1 \dot\cup \Gamma_2$ then $G_\Gamma = G_{\Gamma_1} * G_{\Gamma_2}$
- Every G_Γ embeds as a finite index subgroup of a right-angled Coxeter group

Cohomology of RAAGs

Cohomology of RAAGs

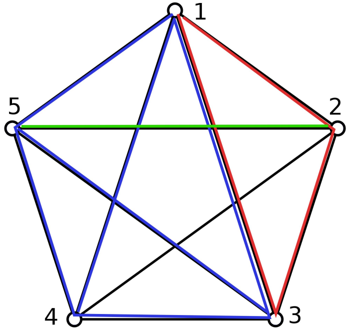
$$H^\bullet(G_\Gamma, \mathbb{F}_p) \simeq \Lambda_\bullet(\Gamma^{op}) := \frac{\Lambda_\bullet\langle\mathcal{V}^*\rangle}{(v_i^* \wedge v_j^* \mid (v_i, v_j) \notin \mathcal{E})}$$

We can identify every product $v_{i_1}^* \cdots v_{i_n}^* \in H^n(G_\Gamma, \mathbb{F}_p)$ with the *clique* (= induced complete subgraph) with vertices $v_{i_1}, \ldots, v_{i_n}$ (so $v_{i_1}^* \cdots v_{i_n}^* = 0$ if $(v_{i_h}, v_{i_k}) \notin \mathcal{E}$ for some i_h, i_k).

The cohomology algebra $H^\bullet(G_\Gamma, \mathbb{F}_p)$ is quadratic, i.e., it is generated by elements of degree 1 and its relations originate from homogeneous relations of degree 2 ($v_i^* v_j^* = -v_j^* v_i^*$, and $v_i^* v_j^* = 0$ if $(v_i, v_j) \notin \mathcal{E}$).

This builds a bridge to Number Theory...

Cohomology of RAAGs
Example 1



The complete graph Γ on 5 vertices $\mathcal{V} = \{v_1, \ldots, v_5\}$

$$H^\bullet(G_\Gamma, \mathbb{F}) \simeq \Lambda_\bullet(\mathcal{V}^*)$$

green $= v_2^* v_5^*$

red $= v_1^* v_2^* v_3^*$

blue $= v_1^* v_3^* v_4^* v_5^*$

$$H^5(G_\Gamma, \mathbb{F}) = \langle v_1^* \cdots v_5^* \rangle$$

$$H^6(G_\Gamma, \mathbb{F}) = 0$$

Cohomology of RAAGs
Example 2

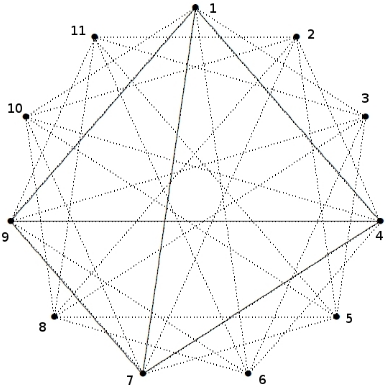


FIGURE 1. Spiga graph

$v_1^* v_2^* = 0$ as $(v_1, v_2) \notin \mathcal{E}$, whereas $v_1^* v_4^* \neq 0$

In the graph we can see $v_1^* v_4^* v_7^* v_9^*$

$H^5(G_\Gamma, \mathbb{F}) = 0$ as there are no 5-cliques in Γ

Pro-p groups

A pro-p group is a *topological* group $\mathcal{G}$ which (1) is compact; (2) is totally disconnected; (3) has a basis $\{N_i \trianglelefteq \mathcal{G}, |\mathcal{G} : N_i| = p^{n_i}\}$ of open neighb.hds of 1. Equivalently, $\mathcal{G} = \varprojlim_i G_i$, with $|G_i| = p^{n_i}$.

Given a group G, its pro-p completion is

$$G_{\hat{p}} = \varprojlim_{N \in \mathcal{N}_p} G/N$$

where $\mathcal{N}_p = \{N \trianglelefteq G, |G : N| = p^{n_N}\}$.

Examples

- finite p-groups
- $\mathbb{Z}_p = \{a_0 + a_1 p + a_2 p^2 + \ldots, a_i \in \mathbb{F}_p\} = \langle 1 \rangle$ is the completion of $\mathbb{Z}$ w.r.t. the topology induced by $\{p^n\mathbb{Z}_p, n \geq 1\}$
- a free pro-p group $\mathcal{F}$ is the completion of F_{abs} w.r.t. the topology induced by $\{U \subseteq F_{\mathrm{abs}} \mid |F_{\mathrm{abs}} : U| = p^n\}$

Galois pro-p groups

Given a field $\mathbb{K}$, the Galois group $\mathcal{G}_{\mathbb{K}}(p) = \mathrm{Gal}(\mathbb{K}(p)/\mathbb{K})$ of the maximal p-extension is the maximal pro-p quotient of the *absolute Galois group* $\mathcal{G}_{\mathbb{K}}$.

Very BIG question in Galois theory

Characterise the $\mathcal{G}_{\mathbb{K}}(p)$'s among all pro-$p$ groups

By the Rost-Voevodsky Theorem ('11), we know that if $1 \neq \sqrt[p]{1} \in \mathbb{K}$ then the $\mathbb{F}_p$-cohomology algebra

$$H^\bullet(\mathcal{G}_{\mathbb{K}}(p), \mathbb{F}_p) = \bigoplus_{n \geq 0} H^n(\mathcal{G}_{\mathbb{K}}(p), \mathbb{F}_p)$$

is quadratic.

Right-angled Artin pro-p groups

Pro-p completion of RAAGs behave almost like abstract RAAGs!

Theorem (Lorensen '10)

A RAAG G_Γ and its pro-p completion $\mathcal{G}_\Gamma$ have the same $\mathbb{F}_p$-cohomology

Thus there are many pro-p groups with quadratic $\mathbb{F}_p$-cohomology! But...

- ... if Γ contains a square as induced subgraph then $\mathcal{G}_\Gamma \not\simeq G_{\mathbb{K}}(p)$ (Q. '14)
- ... if Γ contains a path of 3 edges as induced subgraph then it is conjectured that $\mathcal{G}_\Gamma \not\simeq G_{\mathbb{K}}(p)$ (Weigel and Q.)

Generalised p-RAAGs
and p-graphs

We define a p-graph $\Gamma_f = (\Gamma, f)$ to be a graph $\Gamma = (\mathcal{V}, \mathcal{E})$ endowed with a labelling

$$\begin{gathered} f : \mathcal{E} \to p\mathbb{Z}_p \times p\mathbb{Z}_p, \\ e = (v_i, v_j) \mapsto (f_0(e), f_1(e)), \quad \text{for } i < j \end{gathered}$$

Generalised right-angled Artin pro-p groups

The generalised p-RAAG $\mathcal{G}_\Gamma$ associated to the p-graph $\Gamma_f = (\Gamma, f)$, with $\Gamma = (\mathcal{V}, \mathcal{E})$, is the pro-$p$ group with presentation

$$\mathcal{G}_{\Gamma_f} = \langle v_1, \ldots, v_d \mid [v_i, v_j] = v_i^{f_0(e)} v_j^{f_1(e)}, \forall\, e = (v_i, v_j) \in \mathcal{E} \rangle_{\hat{p}}.$$

Generalised p-RAAGs

Examples

- Let Γ be a graph, let $c \equiv (0,0) \in p\mathbb{Z}_p \times p\mathbb{Z}_p$ be the constant p-labelling on Γ and set $\Gamma_c = (\mathcal{G}, c)$; then $\mathcal{G}_{\Gamma_c}$ is the pro-p completion of an abstract RAAG.

- Let Γ_f be the p-graph

$$\begin{tikzcd} & v_2 \\ v_1 \arrow[ur, "{(\lambda,p^2)}"] \arrow[r, "{(0,0)}"'] & v_3 \end{tikzcd}$$

with $\lambda \in p\mathbb{Z}_p$. Then

$$\mathcal{G}_{\Gamma_f} = \langle v_1, v_2, v_3 \mid [v_1, v_2] = v_1^{\lambda} v_2^{p^2}, [v_1, v_3] = 1 \rangle_{\hat{p}}$$

Generalised p-RAAGs

More examples

- Let Γ_f be the p-graph

$$v_1 \xrightarrow{(a,b)} v_2$$

with $a, b, \lambda \in \mathbb{Z}_p$. Then $\mathcal{G}_{\Gamma_f} = \langle v_1, v_2 \mid [v_1, v_2] = v_1^a v_2^b \rangle_{\hat{p}}$. In fact $\mathcal{G}_{\Gamma_f}$ is a 2-generated *Demushkin group* — i.e., $\mathcal{G}_{\Gamma_f} \simeq \langle x, y \mid [x, y] = y^\lambda, \lambda \in p\mathbb{Z}_p \rangle_{\hat{p}} \simeq \mathbb{Z}_p \rtimes \mathbb{Z}_p$.

- Let Γ_f be the p-graph

$$\begin{tikzcd}
& x_2 \arrow[dr, "{(p,0)}"] & \\
x_1 \arrow[ur, "{(p,0)}"] & & x_3 \arrow[ll, "{(p,0)}"]
\end{tikzcd}$$

Then $\mathcal{G}_{\Gamma_f} = \langle v_1, v_2, v_3 \mid [v_1, v_2]v_1^p = [v_2, v_3]v_2^p = [v_3, v_1]v_3^p \rangle_{\hat{p}}$ is finite.

Cohomology of generalised p-RAAGs

If $H^\bullet(\mathcal{G}_{\Gamma_f},\mathbb{F}_p)$ is quadratic, then $H^\bullet(\mathcal{G}_{\Gamma_f},\mathbb{F}_p)\simeq\Lambda_\bullet(\Gamma^{op})$ (*)

When does this hold?

- Uniform: if $\mathcal{G}_{\Gamma_f}$ is uniform, i.e., Γ is complete and $\mathcal{G}_{\Gamma_f}$ is torsion-free.
- Disjoint union: if Γ_f is the disjoint union of two p-graphs $\Gamma'_{f'},\Gamma''_{f''}$ s.t. (*) holds for $\mathcal{G}_{\Gamma'_{f'}},\mathcal{G}_{\Gamma''_{f''}}$.
- Mirroring: if Γ_f is the p-graph obtained by "mirroring" a p-graph $\Gamma'_{f'}$ s.t. (*) holds for $\mathcal{G}_{\Gamma'_{f'}}$ along a full subgraph of $\Gamma'_{f'}$.
- Complete amalgams: if Γ_f is obtained by gluing two p-graphs $\Gamma'_{f'},\Gamma''_{f''}$ s.t. (*) holds for $\mathcal{G}_{\Gamma_1},\mathcal{G}_{\Gamma_2}$ along a complete full subgraph of $\Gamma'_{f'}$ and $\Gamma''_{f''}$.
- RAAGs: if Γ_c has all labels equal to 0 ($\mathcal{G}_{\Gamma_c}=(G_\Gamma)_{\hat{p}}$).

Cohomology of generalised p-RAAGs
Triangle-free graphs

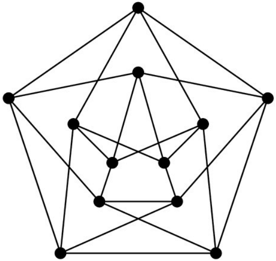
Theorem

Let $\Gamma_f = (\Gamma, f)$ be a p-graph. Then $H^n(\mathcal{G}_{\Gamma_f}, \mathbb{F}_p) = 0$ for $n > 2$ if and only if Γ contains no triangular subgraphs. In this case $H^\bullet(\mathcal{G}_{\Gamma_f}, \mathbb{F}_p)$ is quadratic.

On the other hand, we check which triangular p-graphs Γ_f yield quadratic $H^\bullet(\mathcal{G}_{\Gamma_f}, \mathbb{F}_p)$:

Theorem

A triangular p-graph Γ_f yields quadratic $H^\bullet(\mathcal{G}_{\Gamma_f}, \mathbb{F}_p)$ if and only if $\mathcal{G}_{\Gamma_f}$ is metabelian uniform or $\mathcal{G}_{\Gamma_f} \leq_o \mathrm{SL}_2^1(\mathbb{Z}_p)$ uniform of dimension 3.



A triangle-free graph

A Tits' alternative

For Galois groups of maximal p-extensions one has the following Tits' alternative (Ware '92, Q. '14): if $\mathbb{K} \ni \sqrt[p]{1} \neq 1$, then either $\mathcal{G}_{\mathbb{K}}(p)$ is analytic or it contains a free non-abelian closed subgroup.

Theorem

Let $\mathcal{G}_{\Gamma_f}$ be a generalised p-RAAG with quadratic cohomology. Then either $\mathcal{G}_{\Gamma_f}$ is uniform or it contains a free non-abelian closed subgroup. Moreover, in the latter case $\mathcal{G}_{\Gamma_f}$ is generalised Golod–Shafarevich.

We formulate the following Conjecture: if a pro-p group G has quadratic $\mathbb{F}_p$-cohomology then either G is analytic, or it contains a free non-abelian closed subgroup

THANKS!

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