CSPs of ω-categorical algebras

Thomas Quinn-Gregson
TU Dresden
Joint work with Manuel Bodirsky

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A common framework

Definition

Given a relational structure $(A; \Gamma)$, we define $\mathrm{CSP}(A; \Gamma)$ to be the constraint satisfaction problem (CSP):

- **Instance I:** a finite list of variables V and a finite list of constraints $\mathcal{C}$ in which each constraint is an atomic formula (in the signature of Γ).
- **Question:** Does I have a solution over A? i.e. does there exists a map $V \to A$ satisfying the constraints $\mathcal{C}$?

Example

Given a finite set A, the problem $\mathrm{CSP}(A; \neq)$ has instances of the form

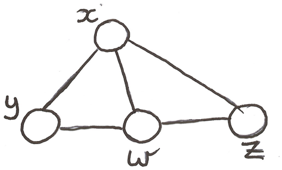
$$x \neq y, y \neq z, y \neq w.$$

The problem is solvable in polynomial time (is **tractable**) if $|A| = 1, 2$, and is $\mathbb{NP}$-complete otherwise.
$\mathrm{CSP}(A; \neq)$ corresponds to graph $|A|$-colouring.

Graph 3-colouring

Instance: $x \neq y,\ x \neq z,\ x \neq w,$
$y \neq w,\ z \neq w.$

Graph:

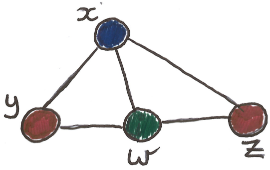


Q^{n}: Can we colour the vertices Red, Blue or Green such that no adjacent vertices are the same colour?

3-Graph colouring

Instance: $x \neq y$, $x \neq z$, $x \neq w$,
$y \neq w$, $z \neq w$.

Graph:



Q^n: Can we colour the vertices
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System of equations satisfactibility

Example (The *system of equations satisfactibility* problem)

Given a finite algebra $(A; F)$, the problem EQN^*_A is:
Input: a system of equations $\mathcal{E}$ over A (constants and variables)
Question: does $\mathcal{E}$ have a solution?

The problem EQN^*_A is equivalent to $\mathrm{CSP}(A; c_1, \ldots, c_n, R_f : f \in F)$ where $A = \{c_1, \ldots, c_n\}$ and $R_f = \{(a_1, \ldots, a_m, f(a_1, \ldots, a_m)) : a_i \in A\}$.

Theorem (Klíma, Tesson, Thérien 2007)

Every CSP over a finite domain is polynomial-time equivalent to EQN^*_S for some finite semigroup S.

ω-categoricity

Much progress has been made in understanding the CSPs of infinite structures: often in the (highly symmetric) ω-categorical setting.

Definition

A structure M is ω-**categorical** if $\mathrm{Th}(M)$ has one countable model, up to isomorphism. Equivalently, if $\mathrm{Aut}(M)$ has only finitely many orbits on its action on M^n for each $n \geq 1$.

Example

A right zero semigroup S has $\mathrm{Aut}(S) = \mathcal{S}_{|S|}$ and is ω-categorical:

- $\forall x, y, z\ [(xy)z = x(yz)]$
- $\forall x, y\ [xy = y]$
- 'correct cardinality'

Example

$\mathrm{CSP}(\mathbb{Q}; <)$ and $\mathrm{CSP}(\mathbb{N}; \neq)$ are tractable (!).

Our Problem

Given an algebra A, we are concerned with the following constraint satisfaction problem, D-EQN$_A$:

- **Instance:** A finite list $\mathcal{E}$ of equations and disequalities over A with variables from a finite set V (no added constants).
- **Question:** Is there an assignment $\phi : V \to A$ such that $\mathcal{E}$ holds in A?

Note that D-EQN$_A$ is equivalent to CSP$(A; \neq, R_f : f \in F)$, where for each basic m-ary function f of A, R_f is the $m+1$-ary relation $\{(a_1, \ldots, a_m, f(a_1, \ldots, a_m)) : a_i \in A\}$.

D-EQN for right zero semigroups

Example

Let S be a right-zero semigroup, so $xy = y$ for each $x, y \in S$. Then an instance of D-EQN$_S$ could be:

$$xy = z, tu = y, x \neq z, t \neq u.$$

Notice this can be satisfied if and only if $y = z, u = y, x \neq z, t \neq u$. The problem is thus equivalent to CSP$(|S|; \neq)$.

Lemma

Let S be a right-zero semigroup. Then D-EQN$_S$ is tractable if $|S| = 1, 2$ or ω, and is $\mathbb{NP}$-complete otherwise.

Motivation

Motivation to study D-EQN$_A$ for an ω-categorical algebra:

- **A non-trivial problem:** As we will see, even in our very restrictive setting we obtain both tractability and hardness.
- **Constraint entailment:** Testing if a list of equations $\mathcal{E}$ implies an equation $u = v$ is equivalent of testing if $\mathcal{E} \cup \{u \neq v\}$ is satisfiable.
- **Links to Identity Checking:** If D-EQN$_A$ is tractable then CHECK-ID(A) is also tractable.
- **Sporadically studied problem:** D-EQN$_A$ has been studied for key structures, including:
  - the lattice reduct of the atomless Boolean algebra $(\mathbb{A}; \cup, \cap)$ ($\mathbb{NP}$-hard, Bodirsky, Hils, Krimkevitch, 2011)
  - the infinite-dimensional vector space over the finite field $\mathbb{F}_q$ (tractable, Bodirsky, Chen, Kára, von Oertzen, 2007).

Polymorphisms

- The hardness of a problem often comes from a lack of symmetry.
- Our usual objects that capture symmetry (automorphism group or endomorphism monoid) are not sufficient.
- We require a more general symmetry - polymorphisms!

Definition

Let $(A;\Gamma)$ be a relational structure. An n-ary homomorphism $f : A^n \to A$ is called a **polymorphism** of A. That is, f preserves every an m-ary relation $\rho \subseteq A^m$:

$$\begin{array}{cccc}
a_{11} & a_{12} & \cdots & a_{1m} \in \rho \\
a_{21} & a_{22} & \cdots & a_{2m} \in \rho \\
\vdots & \vdots & \cdots & \vdots \\
a_{n1} & a_{12} & \cdots & a_{1m} \in \rho \\
\downarrow_f & \downarrow_f & \cdots & \downarrow_f \\
X & X & \cdots & X \in \rho
\end{array}$$

The set of all polymorphisms is denoted $\mathrm{Pol}(A)$.

A simple example

Example

Let $(A; F)$ be an algebra and $\mathcal{A} = (A; \neq, R_f : f \in F)$. Then $f : A^n \to A$ is a polymorphism of $\mathcal{A}$ if and only if f is an algebra homomorphism and

$$x_1 \neq y_1, \ldots, x_n \neq y_n \Rightarrow f(x_1, \ldots, x_n) \neq f(y_1, \ldots, y_n)$$

or, equivalently, if

$$f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n) \Rightarrow x_i = y_i \text{ for some } 1 \leq i \leq n.$$

In particular, every endomorphism of $\mathcal{A}$ is an embedding.

We call a pair of algebras A and B are *bi-embeddable*, denoted $A \equiv B$, if there exists embeddings between them. In this case we have D-EQN$_A$=D-EQN$_B$.

Polymorphisms give tractability

For finite CSPs, the existence of a *Siggers* polymorphism is necessary and sufficient for tractability (Bulatov, Zhuk 2017). For infinite CSPs less is known.

Definition

A 6-ary operation $f \in \mathrm{Pol}(A)$ is called a **pseudo-Siggers** polymorphism if

$$\alpha f(x, y, x, z, y, z) = \beta f(y, x, z, x, z, y)$$

for some unary operations $\alpha, \beta \in \mathrm{Pol}(A)$.

A method by polymorphisms

Theorem (Barto, Pinsker 2006)

If A is ω-categorical and $Pol(A)$ does not contain a pseudo-Siggers polymorphism, then $CSP(A)$ is $\mathbb{NP}$-hard.

This is particularly useful for our problem, and gives rise to classifications of the complexity of key algebras.

Corollary

Let A be an ω-categorical algebra. If D-EQN$_A$ is tractable then there exists a homomorphism $f : A^6 \to A$ such that

- $f(x_1, \ldots, x_6) = f(y_1, \ldots, y_6) \Rightarrow x_i = y_i$ *for some* $1 \leq i \leq 6$,
- *there exists embeddings α and β of A with* $\alpha f(x, y, x, z, y, z) = \beta f(y, x, z, x, z, y)$.

Semilattices

Proposition

Let Y be an ω-categorical semilattice. Then $(Y; \neq, R_\wedge)$ has a pseudo-Siggers if and only if it is bi-embeddable with $Y \times Y$.

- A non-trivial semilattice Y which is bi-embeddable with $Y \times Y$ embeds all finite semilattices.
- Examples include the *universal semilattice*, an ω-categorical semilattice which is the Fraïssé limit of the class of all finite semilattices.

Lemma

Let $\mathbb{U}$ be the universal semilattice. Then D-EQN$_{\mathbb{U}}$ is tractable.

Corollary

Let Y be a non-trivial ω-categorical semilattice. Then D-EQN$_Y$ is tractable if Y is bi-embeddable with the universal semilattice, and is $\mathbb{NP}$-hard otherwise.

Lattices

Similar occurrences holds for lattices:

- A lattice L containing a pseudo-Siggers polymorphism is bi-embeddable with L.
- The universal lattice (which embeds all finite lattices) $\mathbb{L}$ is tractable.

Open: Is a non-trivial lattice L with D-EQN$_L$ necessarily bi-embeddable with $\mathbb{L}$?

Much is now known about the complexity of D-EQN$_A$ for algebras A with $A \equiv A \times A$.

Studied have also been made into the group case (done (?) in the abelian case).

Open: If G is such that D-EQN$_G$ is tractable, then is G abelian?