Local finiteness for Green’s relations in varieties of inverse semigroups and completely regular semigroups

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Locally finite varieties of semigroups

- A **semigroup variety** is the class of all semigroups satisfying some collection of identities (like $xy = yx$ or $x^2 = x$)
- If such a collection can involve only finitely many variables, the variety is of **finite axiomatic rank**
A semigroup variety is the class of all semigroups satisfying some collection of identities (like $xy = yx$ or $x^2 = x$).

If such a collection can involve only finitely many variables, the variety is of finite axiomatic rank.

A variety is locally finite if all its finitely generated members are finite.

For instance, the variety defined by $x^2 = x$ is locally finite while the variety defined by $xy = yx$ is not.
A group $G$ has finite exponent if it satisfies the identity $x^n = 1$ for some integer $n \geq 1$.

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- The bounded Burnside problem asks if there exists an infinite finitely generated group with bounded exponent

- The positive answer was given in 1968 by Novikov and Adian

- We refer to infinite finitely generated groups of finite exponent as Novikov-Adian groups (NAGs)
A semigroup with zero $S$ is a nilsemigroup if some power of each element in $S$ is equal to zero.
A semigroup with zero $S$ is a *nilsemigroup* if some power of each element in $S$ is equal to zero.

A semigroup variety is *periodic* if all its one-generated members are finite.

Clearly, a *locally finite* variety must be periodic.
A remarkable theorem of Mark Sapir

Theorem (Sapir 1987)

A periodic variety $\mathbf{V}$ of semigroups of finite axiomatic rank is **locally finite** if and only if:

(i) all nilsemigroups in $\mathbf{V}$ are locally finite;

(ii) $\mathbf{V}$ contains no NAGs.
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Green’s relations

The following five equivalence relations can be defined on every semigroup $S$:

- $x \mathcal{R} y$ if $x$ and $y$ are prefixes of each other
- $x \mathcal{L} y$ if $x$ and $y$ are suffixes of each other
- $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$
- $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$
- $x \mathcal{J} y$ if $x$ and $y$ are factors of each other
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They were introduced by James Green in 1951 and are collectively referred to as Green’s relations.
Comparison

In general, we have:

\[
\begin{array}{ccc}
 J & D & R \\
 | & | & |
 H & D & L \\
\end{array}
\]
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\[ \mathcal{J} \quad \mathcal{D} \]

\[ \mathcal{R} \quad \mathcal{L} \]

\[ \mathcal{H} \]

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- **Periodic** semigroups: $\mathcal{J} = \mathcal{D}$
- **Groups**: all Green’s relations coincide with the universal relation
Comparison

In general, we have:

- **Periodic** semigroups: $J = D$
- **Groups**: all Green’s relations coincide with the universal relation
- **Nilsemigroups**: all Green’s relations coincide with equality
Let $\mathcal{K}$ be one of the five Green’s relations. A variety $\mathbf{V}$ of semigroups is said to be locally $\mathcal{K}$-finite if each finitely generated semigroup in $\mathbf{V}$ has only finitely many $\mathcal{K}$-classes.
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Volkov, Silva and Soares 2018: classification of locally $\mathcal{K}$-finite semigroup varieties.
Motivation: a natural generalization of local finiteness that bypasses the classical Burnside problem (as all semigroup varieties consisting of groups are locally $\mathcal{K}$-finite for any $\mathcal{K}$).
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Of course, every locally finite variety is locally $\mathcal{K}$-finite for any $\mathcal{K}$; hence, only locally $\mathcal{K}$-finite varieties which are not locally finite are of interest.
(\mathbb{N}, +) has infinitely many $J$-classes

Hence locally $K$-finite varieties are periodic for every $K$
Not really...

- $(\mathbb{N}, +)$ has infinitely many $J$-classes
- Hence locally $K$-finite varieties are **periodic** for every $K$
- Every nilsemigroup is $J$-trivial
- Thus, if $V$ is a locally $K$-finite variety, then nilsemigroups in $V$ are **locally finite**
(\(\mathbb{N}, +\)) has infinitely many \(J\)-classes

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Thus, if \(V\) is a locally \(K\)-finite variety, then nilsemigroups in \(V\) are locally finite

Thus, Sapir’s theorem tells us that every locally \(K\)-finite variety of finite axiomatic rank which is not locally finite contains a NAG
If $\mathcal{K} \subseteq \mathcal{K}'$ are Green’s relations, then every locally $\mathcal{K}$-finite variety of semigroups is locally $\mathcal{K}'$-finite.
Basic properties

- If $\mathcal{K} \subseteq \mathcal{K}'$ are Green's relations, then every locally $\mathcal{K}$-finite variety of semigroups is locally $\mathcal{K}'$-finite.
- Since locally $\mathcal{J}$-finite varieties must be periodic, a variety of semigroups is locally $\mathcal{J}$-finite if and only if it is locally $\mathcal{D}$-finite.
Basic properties

- If $\mathcal{K} \subseteq \mathcal{K}'$ are Green's relations, then every locally $\mathcal{K}$-finite variety of semigroups is locally $\mathcal{K}'$-finite.
- Since locally $\mathcal{J}$-finite varieties must be periodic, a variety of semigroups is locally $\mathcal{J}$-finite if and only if it is locally $\mathcal{D}$-finite.
- A variety of semigroups is locally $\mathcal{H}$-finite if and only if it is both locally $\mathcal{R}$-finite and locally $\mathcal{L}$-finite.
Counterexamples

- No other connections hold in general
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- Let \( n \geq 665 \) be odd and let \( CR_n \) consist of all semigroups which are unions of groups whose exponent divides \( n \). Then \( CR_n \) is locally \( D \)-finite but neither locally \( R \)-finite nor locally \( L \)-finite
Counterexamples

- No other connections hold in general
- Let $n \geq 665$ be odd and let $\text{CR}_n$ consist of all semigroups which are unions of groups whose exponent divides $n$. Then $\text{CR}_n$ is locally $D$-finite but neither locally $R$-finite nor locally $L$-finite
- The variety $\text{LRO}$ of left regular orthogroups is locally $L$-finite but not locally $R$-finite
Results

- We succeeded on characterizing all the locally $\mathcal{K}$-finite varieties of semigroups $\mathbf{V}$ for every Green’s relation $\mathcal{K}$.
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- We succeeded on characterizing all the locally $\mathcal{K}$-finite varieties of semigroups $\mathcal{V}$ for every Green’s relation $\mathcal{K}$
- Forbidden objects are found for each one of the concepts
- Constructions involving NAGs are central in all these results
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Some of the results involve **reductions** to the subvariety \( \text{CR}(V) \) containing all the completely regular semigroups in \( V \).
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The cases of varieties of periodic completely regular semigroups and periodic semigroups with central idempotents were discussed in depth.
Let $G$ be a group and let $H$ be a subgroup of $G$. Denote by $L_H(G)$ the union of $G$ with the set $G_H = \{ gH \mid g \in G \}$ of the left cosets of $H$ in $G$. 
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Denote by $L_H(G)$ the union of $G$ with the set $G_H = \{gH \mid g \in G\}$ of the left cosets of $H$ in $G$.

Extend the multiplication in $G$ to $L_H(G)$ by

\[ g_1(g_2H) = g_1g_2H, \quad (g_1H)g_2 = (g_1H)(g_2H) = g_1H \]
Locally finite extensions

- Note that we view the coset $gH$ as different from $g$ even if $H$ is the trivial subgroup
- $L_H(G)$ is a completely regular semigroup in which $G$ is the group of units and $G_H$ is an ideal consisting of left zeros
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- $L_H(G)$ is a completely regular semigroup in which $G$ is the group of units and $G_H$ is an ideal consisting of left zeros.
- In the dual way, we define the semigroup
  
  $$R_H(G) = G \cup \{Hg \mid g \in G\}$$
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- $L_H(G)$ is a completely regular semigroup in which $G$ is the group of units and $G_H$ is an ideal consisting of left zeros.
- In the dual way, we define the semigroup $R_H(G) = G \cup \{Hg \mid g \in G\}$.
- We say that a semigroup $S$ is a locally finite extension of its ideal $J$ if the Rees quotient $S/J$ is locally finite.
A theorem for $\mathcal{H}$

Theorem (VSS 2018)

A semigroup variety $\mathbf{V}$ of finite axiomatic rank is locally $\mathcal{H}$-finite if and only if either $\mathbf{V}$ is locally finite or satisfies the following conditions:

(i) every semigroup in $\mathbf{V}$ is a locally finite extension of a periodic completely regular ideal;
(ii) $\mathbf{V}$ contains none of the semigroups $L_{\mathcal{H}}(G)$, $R_{\mathcal{H}}(G)$, where $G$ is a NAG and $H$ is its subgroup of infinite index.
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Varieties of semigroups are not enough

- Since \((\mathbb{N}, +)\) is a subsemigroup of \((\mathbb{Z}, +)\) which is not a group, groups do not constitute a variety of semigroups.
Varieties of semigroups are not enough

- Since \((\mathbb{N}, +)\) is a subsemigroup of \((\mathbb{Z}, +)\) which is not a group, groups do not constitute a variety of semigroups.
- But they constitute a variety of unary semigroups, where the unary operation is group inversion.
- Similar problems affect other classes containing groups.
Varieties of unary semigroups

- Two important classes of semigroups (close to groups in different ways) constitute also varieties of unary semigroups:
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  - the variety Inv of inverse semigroups
Varieties of unary semigroups

Two important classes of semigroups (close to groups in different ways) constitute also varieties of unary semigroups:

- the variety $\text{CR}$ of completely regular semigroups
- the variety $\text{Inv}$ of inverse semigroups

Both $\text{CR}$ and $\text{Inv}$ can be seen as $I$-varieties: subvarieties of the variety of $I$-semigroups (defined by the identities $x(yz) = (xy)z$, $(x')' = x$ and $xx'x = x$)
Completely regular semigroups

- The unary operation corresponds to group inversion in the appropriate subgroup
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- $S \in \text{CR}$ is completely simple if $\mathcal{D}$ is the universal relation
Completely regular semigroups

- The unary operation corresponds to group inversion in the appropriate subgroup
- \( S \in \text{CR} \) is completely simple if \( D \) is the universal relation
- Completely simple semigroups can be described as Rees matrix semigroups
Basic facts

- $J = D$ and is a congruence on every $S \in \text{CR}$
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- \( S/D \) is a semilattice and each \( D \)-class is a completely simple semigroup
Basic facts

- $\mathcal{J} = \mathcal{D}$ and is a congruence on every $S \in \text{CR}$
- $S/\mathcal{D}$ is a semilattice and each $\mathcal{D}$-class is a completely simple semigroup
- But every semilattice is locally finite
- Hence $\text{CR}$ is locally $\mathcal{D}$-finite (and consequently locally $\mathcal{J}$-finite)
$L_H(G)$ as a unary semigroup

Let $G$ be a group and let $H$ be a subgroup of $G$.

We make $L_H(G)$ a completely regular unary semigroup by taking inversion in $G$ and $(gH)^{-1} = gH$.
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- Let $G$ be a group and let $H$ be a subgroup of $G$.
- We make $L_H(G)$ a completely regular unary semigroup by taking inversion in $G$ and $(gH)^{-1} = gH$.
- We write $L(G) = L\{1\}(G)$.
- Dually, we define $R(G)$. 

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Local finiteness for Green’s relations
**Theorem (SSV 2019)**

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(i) $\mathbf{V}$ is locally $R$-finite if and only if $\mathbf{V}$ contains none of the completely regular semigroups $L(G)$ where $G$ is either $\mathbb{Z}$ or a NAG.
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(iii) $\mathbf{V}$ is locally $H$-finite if and only if $\mathbf{V}$ contains none of the completely regular semigroups $L(G), R(G)$ where $G$ is either $\mathbb{Z}$ or a NAG.
Inverse semigroups

A semigroup $S$ is inverse if, for every $a \in S$, there exists a unique $a^{-1}$ satisfying $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$.
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Up to isomorphism, inverse semigroups are semigroups of partial injective functions closed under composition and inverse functions.

The unary operation is the obvious one.
Basic facts

In any variety of inverse semigroups:

- locally $J$-finite $\iff$ locally $D$-finite
Basic facts

In any variety of inverse semigroups:

- locally $\mathcal{J}$-finite $\iff$ locally $\mathcal{D}$-finite
- locally $\mathcal{H}$-finite $\iff$ locally $\mathcal{R}$-finite $\iff$ locally $\mathcal{L}$-finite
Adapting a construction of Norman Reilly

- Let $G$ be a nontrivial group
- We define a semigroup with zero $\tilde{N}(G) = G \cup (G \times G) \cup \{0\}$
Adapting a construction of Norman Reilly

1. Let $G$ be a nontrivial group.
2. We define a semigroup with zero $\tilde{N}(G) = G \cup (G \times G) \cup \{0\}$.
3. The multiplication in $G$ is extended to $\tilde{N}(G)$ by:
   \[
   g(h, k) = (gh, k), \quad (h, k)g = (h, kg),
   \]

   \[
   (g, h)(k, \ell) = \begin{cases} 
   (g, \ell) & \text{if } h = k \\
   0 & \text{if } h \neq k
   \end{cases}
   \]
Properties of $\tilde{N}(G)$

- $\tilde{N}(G)$ is an inverse semigroup (with $(g, h)^{-1} = (h, g)$)
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- $\tilde{N}(G)$ has three $\mathcal{D}$-classes
Properties of $\tilde{N}(G)$

- $\tilde{N}(G)$ is an inverse semigroup (with $(g, h)^{-1} = (h, g)$)
- If $G$ is finitely generated, so is $\tilde{N}(G)$
- $\tilde{N}(G)$ has three $D$-classes
- If $G$ is infinite, $\tilde{N}(G)$ has infinitely many $R$-classes
Separating $D$ from $R$

**Theorem (SSV 2019)**

Let $V$ be an $I$-variety of inverse semigroups containing some finitely generated $D$-finite inverse semigroup which is not $R$-finite.
Theorem (SSV 2019)

Let $\mathbf{V}$ be an $I$-variety of inverse semigroups containing some finitely generated $D$-finite inverse semigroup which is not $R$-finite. Then $\mathbf{V}$ contains the bicyclic monoid $B$ or $\tilde{N}(\mathbb{Z})$ or $\tilde{N}(G)$ for some NAG $G$. 
All equivalent!

Theorem (SSV 2019)

The following conditions are equivalent for an $I$-variety $V$ of inverse semigroups:

(i) $V$ is locally $H$-finite;
(ii) $V$ is locally $R$-finite;
(iii) $V$ is locally $L$-finite;
(iv) $V$ is locally $D$-finite;
(v) $V$ is locally $J$-finite.

Note that there exist locally $D$-finite varieties which are not locally finite (e.g. the variety of groups).
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(iv) \( V \) is locally \( D \)-finite;
(v) \( V \) is locally \( J \)-finite.
The following conditions are equivalent for an $I$-variety $\mathbf{V}$ of inverse semigroups:

(i) $\mathbf{V}$ is locally $\mathcal{H}$-finite;
(ii) $\mathbf{V}$ is locally $\mathcal{R}$-finite;
(iii) $\mathbf{V}$ is locally $\mathcal{L}$-finite;
(iv) $\mathbf{V}$ is locally $\mathcal{D}$-finite;
(v) $\mathbf{V}$ is locally $\mathcal{J}$-finite.

Note that there exist locally $\mathcal{D}$-finite varieties which are not locally finite (e.g. the variety of groups).
We need something else

- The preceding results are not enough to provide a characterization by means of forbidden objects.
- Let $W_2$ be the variety of inverse semigroups defined by the identity $x^2 = 0$. 

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- Let $W_2$ be the variety of inverse semigroups defined by the identity $x^2 = 0$.
- With the help of the infinite square-free word due to Morse and Hedlund, we can build a finitely generated $S \in W_2$ with infinitely many $J$-classes.
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- The preceding results are not enough to provide a characterization by means of forbidden objects.
- Let $W_2$ be the variety of inverse semigroups defined by the identity $x^2 = 0$.
- With the help of the infinite square-free word due to Morse and Hedlund, we can build a finitely generated $S \in W_2$ with infinitely many $J$-classes.
- However, $W_2$ does not contain neither $B$ nor $\tilde{N}(\mathbb{Z})$ nor $\tilde{N}(G)$ for any NAG $G$. 
Uniform almost nilsemigroups

A semigroup with zero $S$ is an almost nilsemigroup if some power of each non-idempotent element in $S$ is equal to 0.
Uniform almost nilsemigroups

- A semigroup with zero $S$ is an **almost nilsemigroup** if some power of each non-idempotent element in $S$ is equal to 0.
- $S$ is a **uniform almost nilsemigroup** if such powers can be bounded.
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- $S$ is a uniform almost nilsemigroup if such powers can be bounded.

Theorem (SSV 2019)
Let $\mathcal{V}$ be a variety of inverse semigroups. Then $\mathcal{V}$ is locally $\mathcal{D}$-finite if and only if all uniform almost nilsemigroups in $\mathcal{V}$ are locally finite.
Grazie!  

Thank you!