ON MODULAR ELEMENTS OF THE LATTICE OF SEMIGROUP VARIETIES

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Modular elements

The collection of all semigroup varieties forms a lattice with the following naturally defined operations: for varieties $\mathbf{X}$ and $\mathbf{Y}$, their join $\mathbf{X} \vee \mathbf{Y}$ is the variety generated by the set-theoretical union of $\mathbf{X}$ and $\mathbf{Y}$, while their meet $\mathbf{X} \wedge \mathbf{Y}$ coincides with the set-theoretical intersection of $\mathbf{X}$ and $\mathbf{Y}$.

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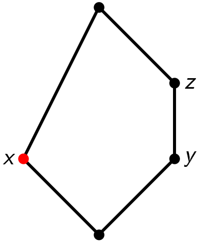
Modular elements

An element x of a lattice L is called *modular* if

$$\forall\, y, z: \quad y \leq z \to (x \vee y) \wedge z = (x \wedge z) \vee y.$$



This configuration is impossible

Modular elements in the lattice **SEM**

Recall that a semigroup variety is called a *nil-variety* if it consists of nilsemigroups or, equivalently, satisfies an identity of the form $x^n \approx 0$ for some natural n.

Proposition 1 (J.Ježek and R.N.McKenzie - 1993; V.Yu.Shaprynskiĭ - 2012)

If $\mathbf{V}$ *is a modular element of the lattice* **SEM** *then either* $\mathbf{V} = \mathbf{SEM}$ *or* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$*, while* $\mathbf{N}$ *is a nil-variety.*

$\mathbf{u} \approx \mathbf{v}$ — *substitutive* if the words $\mathbf{u}$ and $\mathbf{v}$ depend on the same letters and $\mathbf{v}$ may be obtained from $\mathbf{u}$ by renaming of letters

$\mathbf{w} \cdot x \approx x \cdot \mathbf{w} \approx \mathbf{w}$ where the letter x does not occur in the word $\mathbf{w}$ — $\mathbf{w} \approx 0$ — 0-*reduced* identity.

Proposition 2 (B.M.Vernikov - 2007)

A modular nil-variety of semigroups may be given by 0-reduced and substitutive identities only.

Proposition 3 (B.M.Vernikov and M.V.Volkov - 1988; J.Ježek and R.N.McKenzie - 1993)

Every 0-reduced semigroup variety is modular.

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A permutational identity of length 2

A *permutational* identity – an identity of the type

$$x_1 x_2 \ldots x_n \approx x_{1\pi} x_{2\pi} \ldots x_{n\pi}$$

where π is a non-trivial permutation on the set $\{1, 2, \ldots, n\}$.

Proposition 4 (B.M.Vernikov - 2007)

A commutative semigroup variety $\mathbf{V}$ *is a modular element of the lattice* $\mathbf{SEM}$ *if and only if* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$, *while* $\mathbf{N}$ *satisfies the identities* $x^2y = 0$ *and* $xy = yx$.

A *permutational* identity – an identity of the type

$$x_1 x_2 \ldots x_n \approx x_{1\pi} x_{2\pi} \ldots x_{n\pi}$$

where π is a non-trivial permutation on the set $\{1, 2, \ldots, n\}$.

Proposition 5 (D.V.Skokov and B.M.Vernikov - 2019)

A semigroup variety $\mathbf{V}$ *satisfying a permutational identity of length* 3 *is a modular element in the lattice* $\mathbf{SEM}$ *if and only if* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$, *while the variety* $\mathbf{N}$ *satisfies one of the following identity systems:*

a) $xyz = zyx,\ x^2y = 0;$
b) $xyz = yzx,\ x^2y = 0;$
c) $xyz = yxz,\ xyzt = xzty,\ xy^2 = 0;$
d) $xyz = xzy,\ xyzt = yzxt,\ x^2y = 0.$

$$\mathrm{Stab}_{\mathbf{V}}(\mathbf{u}) = \{g \in S_n \mid \mathbf{u} \approx g(\mathbf{u}) \text{ in } \mathbf{V}\}$$

Proposition 6

If a nil-variety of semigroup $\mathbf{V}$ is a modular element of the lattice **SEM** *and* $\mathbf{V}$ *does not satisfy an identity* $\mathbf{u} \approx 0$ *then* $\mathrm{Stab}_{\mathbf{V}}(\mathbf{u})$ *is a modular element of the* $\mathrm{Sub}(S_n)$.

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Proposition 6

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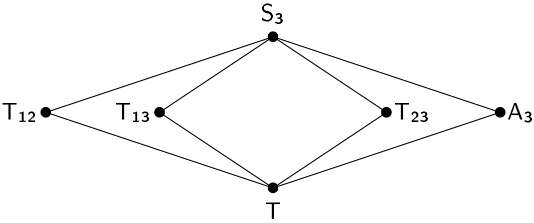
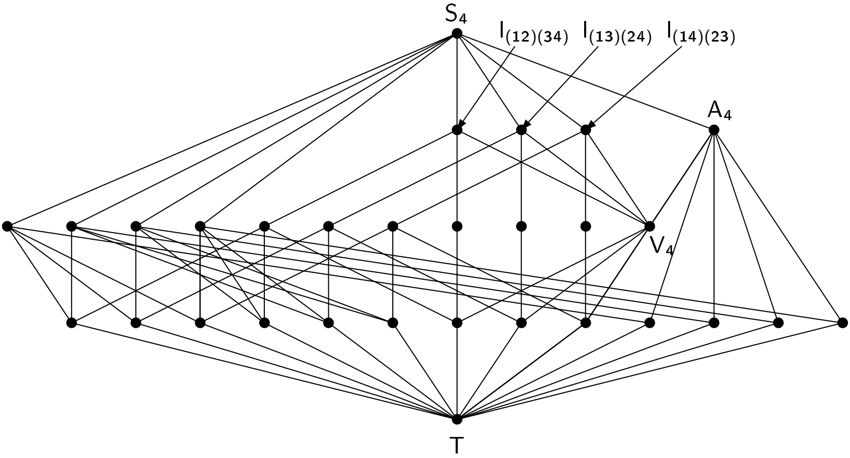
The lattice Sub(S_3)



The lattice Sub(S_4)



$\mathbf{u} \leq \mathbf{v}$ iff $\mathbf{u} = a\xi(\mathbf{v})b$ where $a, b \in F^1$ and $\xi \in \mathrm{End}(F)$

Theorem

A nil-variety of semigroups $\mathbf{V}$ *is a modular element of the lattice* **SEM** *if and only if the following holds:*

a) $\mathbf{V}$ *may be given by 0-reduced and substitutive identities only;*

b) *if* $\mathbf{V}$ *does not satisfy an identity* $\mathbf{u} \approx 0$ *then* $\mathrm{Stab}_{\mathbf{V}}(\mathbf{u})$ *is a modular element of the* $\mathrm{Sub}(S_n)$*;*

c) *there are no uncomparable words* $\mathbf{u}$ *and* $\mathbf{v}$ *such that* $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$ *and*

- $\mathrm{Stab}_{\mathbf{V}}(\mathbf{u}), \mathrm{Stab}_{\mathbf{V}}(\mathbf{v}) \in \{\mathbf{T}_{12}, \mathbf{T}_{23}, \mathbf{T}_{13}\}$;
- $\mathrm{Stab}_{\mathbf{V}}(\mathbf{u}) \in \{\mathbf{T}_{12}, \mathbf{T}_{23}, \mathbf{T}_{13}\}, \mathrm{Stab}_{\mathbf{V}}(\mathbf{v}) = \mathbf{A}_3$;
- $\mathrm{Stab}_{\mathbf{V}}(\mathbf{u}), \mathrm{Stab}_{\mathbf{V}}(\mathbf{v}) \in \{\mathbf{I}_{(12)(34)}, \mathbf{I}_{(13)(24)}, \mathbf{I}_{(14)(23)}\}$;
- $\mathrm{Stab}_{\mathbf{V}}(\mathbf{u}) \in \{\mathbf{I}_{(12)(34)}, \mathbf{I}_{(13)(24)}, \mathbf{I}_{(14)(23)}\}, \mathrm{Stab}_{\mathbf{V}}(\mathbf{v}) = \mathbf{A}_4$

.

Case 1: $n = |\operatorname{con}(\mathbf{u})| \geq 5$

Subcase 1.1: $\operatorname{Stab}_{\mathbf{V}}(\mathbf{u}) = S_n$

Subcase 1.1.1: $\mathbf{u} = x_1 x_2 \dots x_n$

$$\mathbf{V}_{\mathbf{u},S_n} \models \begin{array}{c} x_1 x_2 \dots x_n \approx g(x_1 x_2 \dots x_n) \quad (g \in S_n), \\ x_1^2 x_2 \dots x_{n-1} \approx 0 \end{array}$$

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Subcase 1.1.2: $\mathbf{u} = (x_1 x_2 \dots x_n)^k$

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Case 2: $n = |\operatorname{con}(\mathbf{u})| = 4$

Case 3: $n = |\operatorname{con}(\mathbf{u})| = 3$

Case 4: $n = |\operatorname{con}(\mathbf{u})| = 2$

Case 1: $n = |\mathrm{con}(\mathbf{u})| \geq 5$

Subcase 1.1: $\mathrm{Stab}_{\mathbf{V}}(\mathbf{u}) = S_n$

Subcase 1.2: $\mathrm{Stab}_{\mathbf{V}}(\mathbf{u}) = A_n$

Case 2: $n = |\mathrm{con}(\mathbf{u})| = 4$

Case 3: $n = |\mathrm{con}(\mathbf{u})| = 3$

Case 4: $n = |\mathrm{con}(\mathbf{u})| = 2$

$\mathbf{u} \leq \mathbf{v}$ iff $\mathbf{u} = a\xi(\mathbf{v})b$ where $a, b \in F^1$ and $\xi \in \mathrm{End}(F)$

Theorem

A nil-variety of semigroups $\mathbf{V}$ *is a modular element of the lattice* **SEM** *if and only if the following holds:*

a) $\mathbf{V}$ *may be given by 0-reduced and substitutive identities only;*

b) *if* $\mathbf{V}$ *does not satisfy an identity* $\mathbf{u} \approx 0$ *then* $\mathrm{Stab}_{\mathbf{V}}(\mathbf{u})$ *is a modular element of the* $\mathrm{Sub}(S_n)$*;*

c) *there are no uncomparable words* $\mathbf{u}$ *and* $\mathbf{v}$ *such that* $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$ *and*
   - $\mathrm{Stab}_{\mathbf{V}}(\mathbf{u}), \mathrm{Stab}_{\mathbf{V}}(\mathbf{v}) \in \{\mathbf{T}_{12}, \mathbf{T}_{23}, \mathbf{T}_{13}\}$;
   - $\mathrm{Stab}_{\mathbf{V}}(\mathbf{u}) \in \{\mathbf{T}_{12}, \mathbf{T}_{23}, \mathbf{T}_{13}\}, \mathrm{Stab}_{\mathbf{V}}(\mathbf{v}) = \mathbf{A}_3$;
   - $\mathrm{Stab}_{\mathbf{V}}(\mathbf{u}), \mathrm{Stab}_{\mathbf{V}}(\mathbf{v}) \in \{\mathbf{I}_{(12)(34)}, \mathbf{I}_{(13)(24)}, \mathbf{I}_{(14)(23)}\}$;
   - $\mathrm{Stab}_{\mathbf{V}}(\mathbf{u}) \in \{\mathbf{I}_{(12)(34)}, \mathbf{I}_{(13)(24)}, \mathbf{I}_{(14)(23)}\}, \mathrm{Stab}_{\mathbf{V}}(\mathbf{v}) = \mathbf{A}_4$

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