Quiver presentations for algebras of some order-related monoids of partial functions

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Monoid algebras

- M - finite monoid.
- $\Bbbk$ - field.
- $\Bbbk M$ - monoid algebra.

$$\Bbbk M = \{\sum k_i m_i \mid k_i \in \Bbbk \quad m_i \in M\}$$

- $\Bbbk M$ is usually not a semisimple algebra (even if $\Bbbk = \mathbb{C}$).

Question

Given an interesting monoid M, try to find properties\invariants of $\Bbbk M$

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- $\mathcal{PF}_n \cong \mathcal{F}_{n+1}$ (Umar 1992)

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- Obtaining a Quiver presentation for $\Bbbk\mathcal{PO}_n, \Bbbk\mathcal{PF}_n, \Bbbk\mathcal{PC}_n$. (The main idea).

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$\mathbb{k}$-linear Category presentation

Let E be a category, we can define the **linearization** $L_{\mathbb{k}}[E]$. The categories $L_{\mathbb{k}}[E]$ and E have the same objects, and every hom-set $L_{\mathbb{k}}[E](a, b)$ is the $\mathbb{k}$-vector space with basis $E(a, b)$. Composition is defined naturally.

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(Neglected Issues: Precise definition of the quiver (generators). Morita Equivalence.)

Big picture

Category

$\mathbb{k}$ - linear category

Monoid

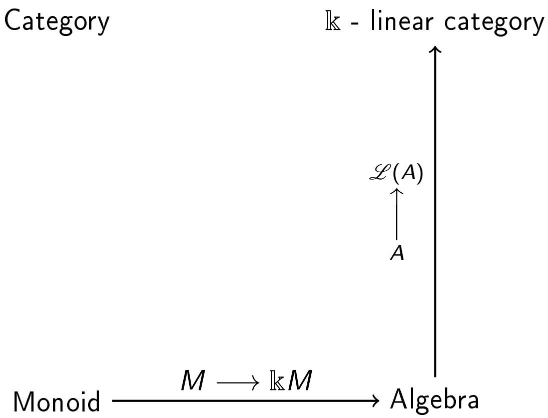
Algebra

Big picture

Category $\quad\quad$ $\mathbb{k}$ - linear category

Monoid $\xrightarrow{M \longrightarrow \mathbb{k}M}$ Algebra

Big picture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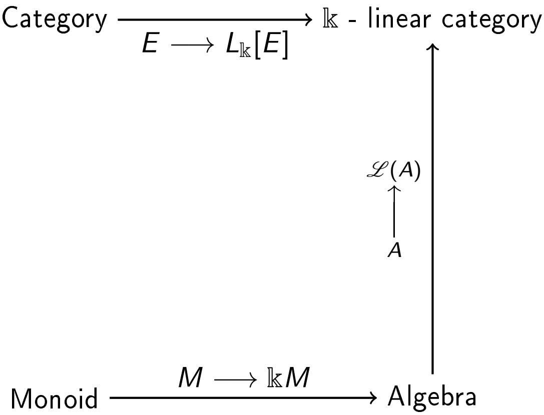
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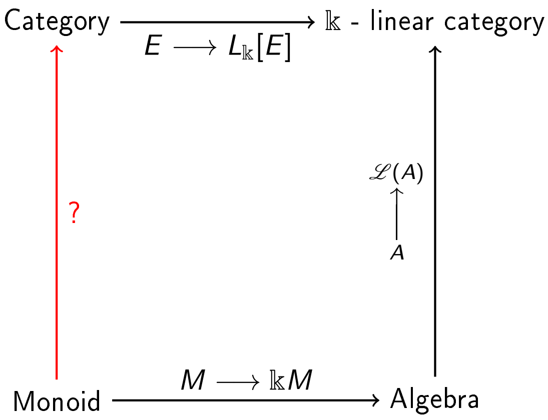

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Proposition (IS 2016)

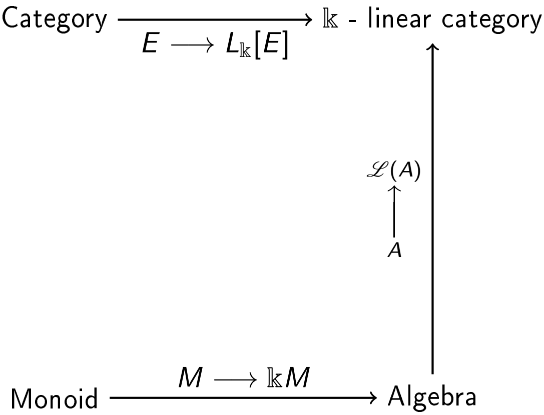
There is an isomorphism of algebras

$$\Bbbk\mathcal{PO}_n \simeq \Bbbk\mathcal{EO}_n, \quad \Bbbk\mathcal{PF}_n \simeq \Bbbk\mathcal{EF}_n, \quad \Bbbk\mathcal{PC}_n \simeq \Bbbk\mathcal{EC}_n$$

Remark

Similar result holds for many other finite semigroups.(Solomon, Steinberg, Guo,Chen, IS, Wang)

Big picture #2



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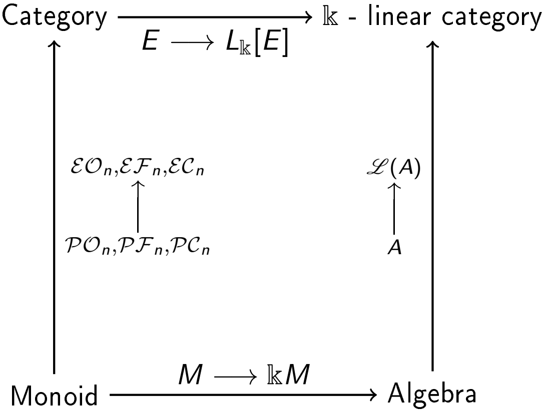

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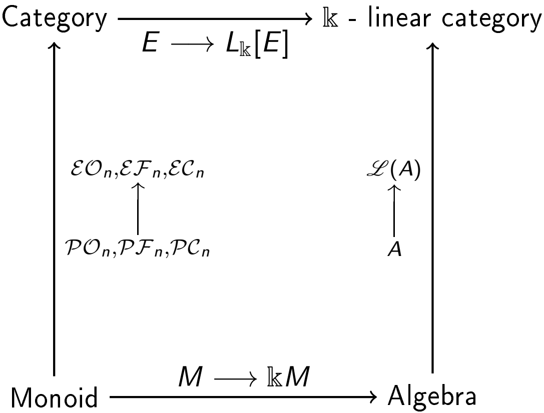

Proposition

This diagram is "commutative" for $\mathcal{PO}_n$, $\mathcal{PF}_n$ and $\mathcal{PC}_n$.

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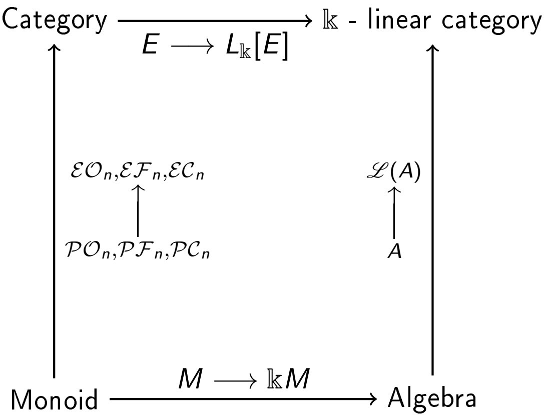

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$L_{\Bbbk}[\mathcal{EO}_n] \simeq \mathscr{L}(\Bbbk\mathcal{PO}_n)$, $L_{\Bbbk}[\mathcal{EF}_n] \simeq \mathscr{L}(\Bbbk\mathcal{PF}_n)$, $L_{\Bbbk}[\mathcal{EC}_n] \simeq \mathscr{L}(\Bbbk\mathcal{PC}_n)$.

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Lemma

If $\langle Q \mid R \rangle$ is a (category) presentation of E then it is also a ($\mathbb{k}$- linear category) presentation of $L_{\mathbb{k}}[E]$.

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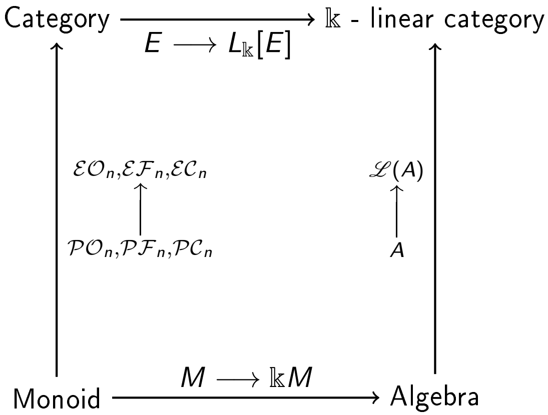

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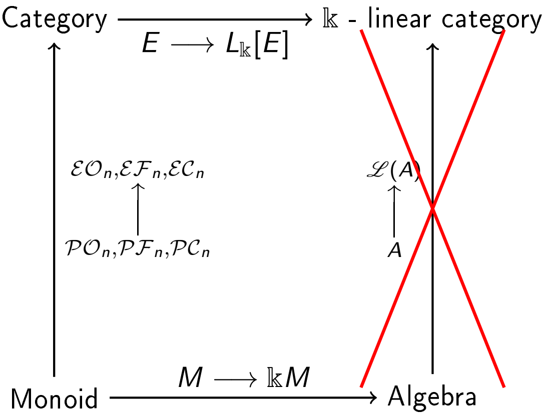

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Remark

Using this approach we can obtain also a description of other invariants such as the Cartan matrix, Loewy length etc..

Thank you!

Appendix: Results #1

Theorem (IS)

Let $M = \mathcal{PO}_n$ *. Denote by* d_i^k *the morphism corresponding to the unique order preserving onto function* $f : [k+1] \to [k]$ *such that* $f(i) = f(i+1)$ *then*

- $d_i^{k-1} d_j^k = d_{j-1}^{k-1} d_i^k \quad (2 \leq k \leq n-1, \quad 1 \leq i < j \leq k)$

is a quiver presentation for $\Bbbk \mathcal{PO}_n$.

Theorem (IS)

Let$M = \mathcal{PC}_n$*. Denote by* d_i^A *the morphism whose domain is* A *and* $i \in A$ *is its unique element such that* $d_i^A(i) = i - 1$*. Then*

- $d_i^{A_j} d_j^A = d_j^{A_i} d_i^A \quad (j > i+1, \quad i, j \in A)$
- $d_i^{A_{i+1}} d_{i+1}^A = d_i^{(A_i)_{i+1}} d_{i+1}^{A_i} d_i^A \quad (i, i+1 \in A)$

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Appendix: Results #2

Theorem (IS)

Let $M = \mathcal{PF}_n$. Denote by $d^A_{i,j}$ is the morphism whose domain is A and $j \in A$ is its unique element such that $d^A_{i,j}(j) = i \neq j$. Then

- $d^{A_{s,t}}_{i,j} d^A_{s,t} = d^{A_{i,j}}_{s,t} d^A_{i,j} \quad (s > j, \quad t, j \in A)$
- $d^{A_{s,t}}_{i,j} d^A_{s,t} = d^{A_{i,t}}_{i,j} d^{A_{i,j}}_{j,t} d^A_{i,j} \quad (s = j, \quad t, j \in A)$
- $d^{A_{s,t}}_{i,j} d^A_{s,t} = d^{A_{i,t}}_{s,j} d^{A_{i,j}}_{j,t} d^A_{i,j} \quad (i < s < j, \quad s, t, j \in A)$
- $d^{A_{s,t}}_{i,j} d^A_{s,t} = d^{A_{i,t}}_{s,j} d^{A_{i,j}}_{j,t} d^A_{i,j} \quad (s \leq i, \quad t, j \in A)$
- $d^{A_{s,t}}_{i,j} d^A_{s,t} = d^{A_{i,t}}_{s,j} d^{A_{i,j}}_{j,t} d^{A_{s,j}}_{i,s} d^A_{s,j} \quad (i < s < j, \quad t, j \in A, \quad s \notin A)$
- $d^{(A_{i,t})_{s,i}}_{i,j} d^{A_{i,t}}_{s,i} d^A_{i,t} = d^{A_{i,t}}_{s,j} d^{A_{i,j}}_{j,t} d^A_{i,j} \quad (s < i < j < t, \quad t, j \in A, \quad i \notin A)$

is a quiver presentation of $\Bbbk\mathcal{PF}_n$.