Generated groups, Cayley color digraphs, and related cardinal metrics

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Generated groups and word metrics

Recall that a generated group (G, S) consists of a group G and a specific generating set S.

Definition 1 (Word metrics)

Let (G, S) be a generated group. The word metric with respect to S, denoted by d_W, is defined on G by

$$d_W(g, h) = \min\{n \in \mathbb{N}\colon\ g^{-1}h = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_n^{\epsilon_n}, s_i \in S, \epsilon_i \in \{\pm 1\}\} \tag{1}$$

for all $g, h \in G$ with $g \neq h$ and $d_W(g, g) = 0$ for all $g \in G$.

Group norms

Let (G, S) be a generated group. Then

$$\|g\|_W = \begin{cases} \min\{n \in \mathbb{N} \colon g = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_n^{\epsilon_n}, s_i \in S, \epsilon_i \in \{\pm 1\}\} & \text{if } g \neq e, \\ 0 & \text{if } g = e. \end{cases}$$

defines a *group-norm* on G.

Furthermore, we have

$$d_W(g, h) = \|g^{-1} h\|_W \tag{2}$$

for all $g, h \in G$.

Cayley digraphs and graphs

Let (G, S) be a generated group. Recall that the Cayley digraph of G with respect to S, denoted by $\overrightarrow{\mathrm{Cay}}(G, S)$, is a digraph such that

1. the vertex set is G;
2. the set of arcs is $\{(g, gs) : g \in G, s \in S\}$.

The (undirected) Cayley graph of G with respect to S, denoted by $\mathrm{Cay}\,(G, S)$, is defined as the underlying graph of $\overrightarrow{\mathrm{Cay}}(G, S)$:

1. the vertex set is G;
2. $\{g, h\}$ is an edge in $\mathrm{Cay}\,(G, S)$ if and only if (g, h) or (h, g) is an arc in $\overrightarrow{\mathrm{Cay}}(G, S)$.

Word metrics and Cayley graphs

Theorem 2 (Well-known Result)

Let (G, S) be a generated group. For all $g \neq h$ in G, $d_W(g, h)$ equals the shortest length of a path in $\mathrm{Cay}(G, S)$ connecting vertices g and h.

Cardinal norms

Let (G,S) be a generated group. Define a function $\|\cdot\|$ corresponding to S, called the cardinal norm on G, by

$$\|g\| = \min\{|A| : A \subseteq S \text{ and } g \in \langle A\rangle\} \tag{3}$$

for all $g \in G$.

Theorem 3

Let (G,S) be a generated group. Then $\|\cdot\|$ is a group-norm:

1. $\|g\| \geq 0$ for all $g \in G$ and $\|g\| = 0$ if and only if g is the identity of G;
2. $\|g^{-1}\| = \|g\|$ for all $g \in G$;
3. $\|gh\| \leq \|g\| + \|h\|$ for all $g, h \in G$.

Cardinal metrics

Let (G, S) be a generated group. The cardinal metric on G is defined by

$$d_C(g, h) = \|g^{-1}h\| \tag{4}$$

for all $g, h \in G$.

Some properties of the cardinal metric are listed below.

1. $d_C(g, h) \leq |S|$
2. $d_C(g, h) \leq d_W(g, h)$
3. The following are classes of isometries of (G, d_C):
   - the left multiplication maps $L_g\colon h \mapsto gh$;
   - the automorphisms τ of G with the property that $\|\tau(g)\| = \|g\|$ for all $g \in G$;
   - the automorphisms τ of G with the property that $\tau(S) = S$.

Comparison between word and cardinal metrics

Theorem 4

Let (G, S) be a *finitely* generated group.

1. If (G, d_W) is of *infinite* diameter, then (G, d_W) and (G, d_C) are not quasi-isometric.
2. If (G, d_W) is of *finite* diameter, then (G, d_W) and (G, d_C) are bi-Lipschitz equivalent.

Characterization of cardinal metrics

Let (G, S) be a generated group. Let $\overrightarrow{\mathrm{Cay}}_c(G, S)$ be the Cayley color digraph of G with respect to S.

Theorem 5[1]

If g and h are distinct elements of G, then $d_C(g, h)$ equals the minimum number of colors associated with a path in $\overrightarrow{\mathrm{Cay}}_c(G, S)$ connecting vertices g and h.

[1] TS, *Geometry of generated groups with metrics induced by their Cayley color graphs*, Analysis and Geometry in Metric Spaces, **7** (2019), pp. 15–21

An example: The quaternion group Q_8

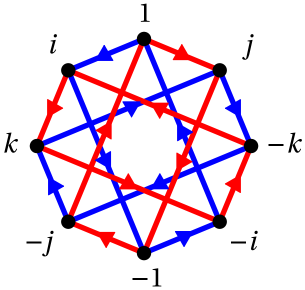

Figure: $\overrightarrow{\mathrm{Cay}}_c(Q_8, \{i, j\})$; blue arcs are induced by i and red arcs are induced by j.

Color-permuting and color-preserving automorphisms

A graph automorphism α of $\overrightarrow{\mathrm{Cay}}_c(G, S)$ is called a color-permuting automorphism of $\overrightarrow{\mathrm{Cay}}_c(G, S)$ if there exists a permutation $\sigma \in \mathrm{Sym}(S)$ such that (g, h) has color c if and only if $(\alpha(g), \alpha(h))$ has color $\sigma(c)$ for all g, h in G.

A graph automorphism α of $\overrightarrow{\mathrm{Cay}}_c(G, S)$ is called a color-preserving automorphism of $\overrightarrow{\mathrm{Cay}}_c(G, S)$ if (g, h) has color c if and only if $(\alpha(g), \alpha(h))$ has color c for all g, h in G.

A class of isometries of (G, d_C)

Theorem 6

If (G, S) is a generated group, then the color-permuting automorphisms of $\overrightarrow{\mathrm{Cay}}_c(G, S)$ are isometries of G with respect to the cardinal metric.

Corollary 7

If (G, S) is a generated group, then the color-preserving automorphisms of $\overrightarrow{\mathrm{Cay}}_c(G, S)$ are isometries of G with respect to the cardinal metric.