Algorithmic properties of tree-like inverse monoids

Nóra Szakács
(joint work with Robert Gray and Pedro Silva)

University of York, UK

SandGAL 2019

Inverse monoids

Definition

A monoid M is called an inverse monoid if every element $m \in M$ has a unique inverse m^{-1} satisfying

$$mm^{-1}m = m, m^{-1}mm^{-1} = m^{-1}.$$

Inverse monoids

Definition

A monoid M is called an inverse monoid if every element $m \in M$ has a unique inverse m^{-1} satisfying

$$mm^{-1}m = m, m^{-1}mm^{-1} = m^{-1}.$$

The typical example: the symmetric inverse monoid on a set X: $X \to X$ partial injective maps under partial multiplication.

Inverse monoids

Definition

A monoid M is called an inverse monoid if every element $m \in M$ has a unique inverse m^{-1} satisfying

$$mm^{-1}m = m, m^{-1}mm^{-1} = m^{-1}.$$

The typical example: the symmetric inverse monoid on a set X: $X \to X$ partial injective maps under partial multiplication.

Natural partial order: $a \leq b$ iff there exists an idempotent e with $a = be$.

Inverse monoid presentations

An inverse monoid presentation: $M = Inv\langle A \mid u_i = v_i \ (i \in I)\rangle$, where u_i, v_i are words in $(A \cup A^{-1})^*$
— the "most general" inverse monoid generated by A, where $u_i = v_i$.

Inverse monoid presentations

An inverse monoid presentation: $M = Inv\langle A \mid u_i = v_i \ (i \in I)\rangle$, where u_i, v_i are words in $(A \cup A^{-1})^*$
— the "most general" inverse monoid generated by A, where $u_i = v_i$.

The **word problem** for M: given $u, v \in (A \cup A^{-1})^*$, do we have $u =_M v$?

Inverse monoid presentations

An inverse monoid presentation: $M = Inv\langle A \mid u_i = v_i \ (i \in I)\rangle$, where u_i, v_i are words in $(A \cup A^{-1})^*$
— the "most general" inverse monoid generated by A, where $u_i = v_i$.

The **word problem** for M: given $u, v \in (A \cup A^{-1})^*$, do we have $u =_M v$?

M has **solvable word problem** if there exists an algorithm that decides the word problem.

Inverse monoid presentations

An inverse monoid presentation: $M = Inv\langle A \mid u_i = v_i \ (i \in I)\rangle$, where u_i, v_i are words in $(A \cup A^{-1})^*$
— the "most general" inverse monoid generated by A, where $u_i = v_i$.

The **word problem** for M: given $u, v \in (A \cup A^{-1})^*$, do we have $u =_M v$?

M has **solvable word problem** if there exists an algorithm that decides the word problem.

The main result of the talk: inverse monoids which satisfy a certain geometric property have solvable word problem, (and other nice algorithmic properties).

The Cayley graph

Let M be an inverse monoid generated by A.
The Cayley graph $\Gamma(M, A)$ of M is an edge-labeled, directed graph

- with vertex set M,
- for any $m \in M$, and any $a \in A \cup A^{-1}$, $m \xrightarrow{a} ma$ is an edge.

The Cayley graph

Let M be an inverse monoid generated by A.
The Cayley graph $\Gamma(M, A)$ of M is an edge-labeled, directed graph

- with vertex set M,
- for any $m \in M$, and any $a \in A \cup A^{-1}$, $m \xrightarrow{a} ma$ is an edge.

Note:

- aa^{-1}, $a^{-1}a$ are not always loops,

The Cayley graph

Let M be an inverse monoid generated by A.
The Cayley graph $\Gamma(M, A)$ of M is an edge-labeled, directed graph

- with vertex set M,
- for any $m \in M$, and any $a \in A \cup A^{-1}$, $m \xrightarrow{a} ma$ is an edge.

Note:

- aa^{-1}, $a^{-1}a$ are not always loops,
- the Cayley graph is not strongly connected.

The strongly connected components

Fact: if m, ma are in the strongly connected component, then $maa^{-1} = m$, that is, in these components, edges occur in inverse pairs.

The strongly connected components

Fact: if m, ma are in the strongly connected component, then $maa^{-1} = m$, that is, in these components, edges occur in inverse pairs.

Definition

The strongly connected component of m is called the **Schützenberger graph** of m, and is denoted by $\mathcal{S}(m)$.

The strongly connected components

Fact: if m, ma are in the strongly connected component, then $maa^{-1} = m$, that is, in these components, edges occur in inverse pairs.

Definition

The strongly connected component of m is called the **Schützenberger graph** of m, and is denoted by $\mathcal{S}(m)$.

Fact: mm^{-1} is always a vertex of $\mathcal{S}(m)$, moreover, it is the unique idempotent vertex.

Schützenberger automata

Let $M = \langle A \rangle$ be an inverse monoid.
The Schützenberger automaton of m: $\mathcal{SA}(m) = (\mathcal{S}(m), mm^{-1}, m)$.

Theorem (Stephen, 1990)

- *$L(\mathcal{SA}(m)) = \{w \in (A \cup A^{-1})^* : w \geq_M m\}$,*
- *for $u, v \in (A \cup A^{-1})^*$, $u_M = v_M$ iff $v \in L(\mathcal{SA}(u))$ and $u \in L(\mathcal{SA}(v))$,*
- *the word problem for M boils down to deciding the languages of the Schützenberger automata*

Constructing $\mathcal{SA}(w)$

Let $M = \mathrm{Inv}\langle A \mid u_i = v_i \ (i \in I)\rangle$, and $w \in (A \cup A^{-1})^*$.
To build $\mathcal{SA}(w)$,

1. start with the linear automaton corresponding to w;
2. **expand:** if one side of a relation is readable between two vertices and the other is not, add it to the graph;
3. **fold:** if $u \xrightarrow{a} v_1$ and $u \xrightarrow{a} v_2$, or $v_1 \xrightarrow{a} u$ and $v_2 \xrightarrow{a} u$, identify v_1 and v_2;
4. repeat (2) and (3) until the graph stabilizes (possibly forever).

Constructing $\mathcal{SA}(w)$

Let $M = \mathrm{Inv}\langle A \mid u_i = v_i \ (i \in I)\rangle$, and $w \in (A \cup A^{-1})^*$.
To build $\mathcal{SA}(w)$,

1. start with the linear automaton corresponding to w;
2. **expand:** if one side of a relation is readable between two vertices and the other is not, add it to the graph;
3. **fold:** if $u \xrightarrow{a} v_1$ and $u \xrightarrow{a} v_2$, or $v_1 \xrightarrow{a} u$ and $v_2 \xrightarrow{a} u$, identify v_1 and v_2;
4. repeat (2) and (3) until the graph stabilizes (possibly forever).

Theorem (Stephen)

These operations are confluent, and the limit automaton is $\mathcal{SA}(w)$.

Graphs as metric spaces

Let Γ be a graph, $u, v \in V(\Gamma)$,

$$d(u, v) := \min\{n : e_1 \dots e_n \text{ is a path from } u \text{ to } v\}.$$

This is a **metric** on the vertices.

Graphs as metric spaces

Let Γ be a graph, $u, v \in V(\Gamma)$,

$$d(u,v) := \min\{n : e_1 \dots e_n \text{ is a path from } u \text{ to } v\}.$$

This is a **metric** on the vertices.

If Γ is a digraph, it may not be symmetric, however for Schützenberger graphs, it is.

Graphs as metric spaces

Let Γ be a graph, $u, v \in V(\Gamma)$,

$$d(u, v) := \min\{n : e_1 \dots e_n \text{ is a path from } u \text{ to } v\}.$$

This is a **metric** on the vertices.

If Γ is a digraph, it may not be symmetric, however for Schützenberger graphs, it is.

A **geodesic** from u to v: a path with length $d(u, v)$.

Tree-like graphs

Definition

A graph is tree-like if it is **quasi-isometric** to a tree.

Tree-like graphs

Definition

A graph is tree-like if it is **quasi-isometric** to a tree.

Definition

An inverse monoid is tree-like if all its Schützenberger graphs are tree-like (this is independent of the generating system).

Tree-like graphs

Definition

A graph is tree-like if it is **quasi-isometric** to a tree.

Definition

An inverse monoid is tree-like if all its Schützenberger graphs are tree-like (this is independent of the generating system).

Examples

- finite inverse monoids
- free inverse monoids (Schützenberger graphs are finite trees)
- $\mathrm{Inv}\langle A \mid u_i = v_i \ (i \in I)\rangle$ where u_i, v_i are equal to 1 in the free group (Schützenberger graphs are trees)
- virtually free groups
- $\mathrm{Inv}\langle x_1, \ldots, x_n, y_1, \ldots, y_n \mid [x_1, y_1] \cdot \ldots \cdot [x_n, y_n] = 1\rangle \ (n \geq 2)$

Regular geodesics

Let $M = \langle X \rangle$ be an inverse monoid.
$\mathrm{Geo}(w) :=$ set of labels of geodesics in $\mathcal{S}(w)$ from $x_0 := ww^{-1}$.

Theorem

If M is a finitely presented tree-like inverse monoid, then $\mathrm{Geo}(w)$ *is regular.*

Regular geodesics

Let $M = \langle X \rangle$ be an inverse monoid.
$\mathrm{Geo}(w) :=$ set of labels of geodesics in $\mathcal{S}(w)$ from $x_0 := ww^{-1}$.

Theorem

If M is a finitely presented tree-like inverse monoid, then $\mathrm{Geo}(w)$ *is regular.*

Remark: there exist f.g. tree-like inverse monoids which don't have a regular set of geodesics, and hence are not finitely presented.

The main theorems

Theorem

If M is a finitely presented inverse monoid, and $\mathcal{S}(w)$ is tree-like, then $L(\mathcal{S}\mathcal{A}(w))$ is context-free.

The main theorems

Theorem

If M is a finitely presented inverse monoid, and $\mathcal{S}(w)$ is tree-like, then $L(\mathcal{S}\mathcal{A}(w))$ is context-free.

Moreover, there is an algorithm that constructs the pushdown automata with input the presentation and w.

The main theorems

Theorem

If M is a finitely presented inverse monoid, and $\mathcal{S}(w)$ is tree-like, then $L(\mathcal{S}\mathcal{A}(w))$ is context-free.

Moreover, there is an algorithm that constructs the pushdown automata with input the presentation and w. Hence:

Theorem

The word problem for tree-like inverse monoids is uniformly decidable, that is, there is a Turing machine with input

- $M = \mathrm{Inv}\langle A \mid u_i = v_i \ (1 \leq i \leq n)\rangle$
- $u, v \in (A \cup A^{-1})^*$

that halts if and only if M is tree-like, and then decides if $u =_M v$.

Thank you for your attention!