Finding generating sets of finite groups using their Cayley graphs

Presented by
Tanakorn Udomworarat

Advisor
Teerapong Suksumran

Department of Mathematics, Faculty of Science,
Chiang Mai University, Thailand

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- Equivalence relation and components of Cayley graphs
- Connections between components and cosets
- Construction generating sets of groups
- Finding minimal generating sets of finite groups

Cayley graphs

Cayley digraph $\overrightarrow{\mathrm{Cay}}(G, A)$

- G is a group
- $A \subseteq G$
- Vertex set is G
- Arc set is $\{(g, ga) : g \in G, a \in A\}$

Cayley color digraph $\overrightarrow{\mathrm{Cay}}_c(G, A)$

- Arc (g, ga) is given color a

Cayley graph $\mathrm{Cay}(G, A)$

- Underlying graph of $\overrightarrow{\mathrm{Cay}}(G, A)$
- $\{u, v\}$ is an edge if and only if $u = va$ or $v = ua$ for some $a \in A$
- $\{u, v\}$ is an edge in $\mathrm{Cay}(G, A)$ if and only if (u, v) or (v, u) is an arc in $\overrightarrow{\mathrm{Cay}}(G, A)$

Equivalence relation and components of Cayley graphs

Definition 1

The **equivalence relation** $\mathfrak{p}$ on G is defined by $u \, \mathfrak{p} \, v$ if and only if either $u = v$ or there is a path from u to v in $\mathrm{Cay}(G, A)$.

Definition 2

C is a **component** of $\mathrm{Cay}\,(G, A)$ if and only if there is an equivalence class X of the relation $\mathfrak{p}$ such that C is the subgraph of $\mathrm{Cay}\,(G, A)$ induced by X, that is, $C = \mathrm{Cay}\,(G, A)[X]$.

Remark 3

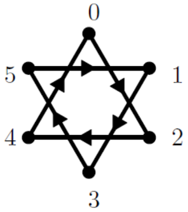
- $u \, \mathfrak{p} \, v$ if and only if u and v are in the same component of $\mathrm{Cay}\,(G, A)$.

Relations between components and cosets

Theorem 4

Let $u, v \in G$. Then u and v are in the same coset of $\langle A \rangle$ in G if and only if $u \, \mathfrak{p} \, v$, where $\mathfrak{p}$ is the equivalence relation induced by $\mathrm{Cay}(G, A)$.

Example In $\mathbb{Z}_6$, $\langle 2 \rangle = \{0, 2, 4\}$.
Cosets of $\langle 2 \rangle$ in $\mathbb{Z}_6$ are $0 + \langle 2 \rangle = \{0, 2, 4\}$ and $1 + \langle 2 \rangle = \{1, 3, 5\}$.



$\overrightarrow{\mathrm{Cay}}\,(\mathbb{Z}_6, \{2\})$

Connections between components and cosets

Corollary 5

Let $X \subseteq G$. Then X is an equivalence class of the relation $\mathfrak{p}$ induced by $\mathrm{Cay}(G, A)$ if and only if X is a coset of $\langle A \rangle$ in G.

Corollary 6

Let G be a group and let $A \subseteq G$. Any component of $\mathrm{Cay}(G, A)$ is of the form

$$\mathrm{Cay}(G, A)[g\langle A \rangle]$$

for some $g \in G$.

Corollary 7

The number of components of $\mathrm{Cay}\,(G, A)$ equals $[G : \langle A \rangle]$, the index of $\langle A \rangle$ in G.

Construction generating sets of groups

Generating set of a group G is a subset such that each element of the group can be expressed as the combination (under the group operation) of finitely many elements of the subset and their inverses.

Theorem 8 (Well-known Result)

The Cayley graph $\mathrm{Cay}\,(G, A)$ is connected if and only if $G = \langle A \rangle$.

Theorem 9

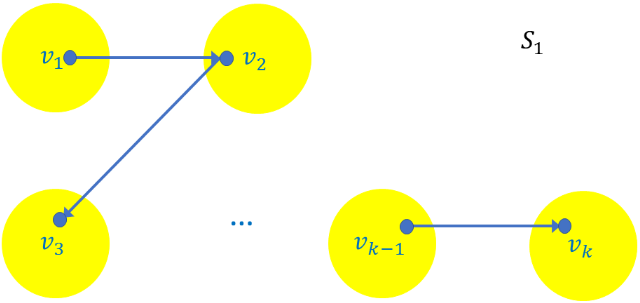
If $\mathrm{Cay}\,(G, A)$ has finitely many components and if $v_1, v_2, \ldots, v_k$ are representative vertices in all the components of $\mathrm{Cay}\,(G, A)$, then

$$S_1 = A \cup \{v_1^{-1}v_2, v_2^{-1}v_3, \ldots, v_{k-1}^{-1}v_k\} \quad \text{and} \quad S_2 = A \cup \{v_1^{-1}v_2, v_1^{-1}v_3, \ldots, v_1^{-1}v_k\}$$

form generating sets of G.

Construction generating sets of groups

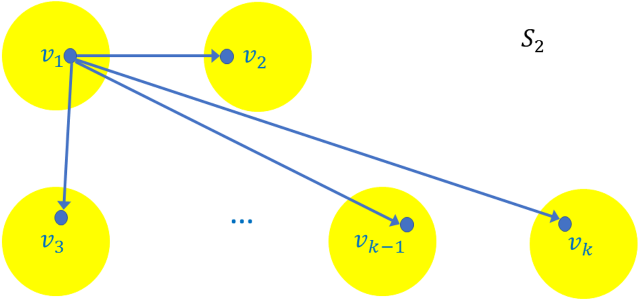
Idea of Theorem 9



Simple diagram of $\overrightarrow{\mathrm{Cay}}(G, S_1)$; $S_1 = A \cup \{v_1^{-1}v_2, v_2^{-1}v_3, \ldots, v_{k-1}^{-1}v_k\}$

Construction generating sets of groups

Idea of Theorem 9



Simple diagram of $\overrightarrow{\mathrm{Cay}}(G, S_2)$; $S_2 = A \cup \{v_1^{-1}v_2, v_1^{-1}v_3, \ldots, v_1^{-1}v_k\}$

Finding minimal generating sets of finite groups

A generating set A of a group G is **minimal** if no proper subset of A generates G.

Let G be a finite group.

Algorithm for finding minimal generating sets

1. Set $A = \{a_1\}$, where $a_1 \in G$ and $a_1 \neq e$.

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7. If $i = 1$, stop. Otherwise, set $i = i - 1$.
8. Draw $\mathrm{Cay}(G, A \backslash \{a_i\})$.

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8. Draw $\mathrm{Cay}\,(G, A \backslash \{a_i\})$.
9. If $\mathrm{Cay}\,(G, A \backslash \{a_i\})$ is connected, set $A = A \backslash \{a_i\}$. Otherwise, go to step (10).

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10. Return to step (7).

When this algorithm stops, we have A is a minimal generating set of G.

Finding minimal generating sets of finite groups

Let G be the group defined by presentation

$$G = \langle a, b, c\colon a^2 = b^2 = (ab)^2 = c^3 = acabc^{-1} = abcbc^{-1} \rangle. \tag{1}$$

Its Cayley table is given by this table [1].

$\cdot$	e	a	b	ab	c	ac	bc	abc	cc	acc	bcc	$abcc$
e	e	a	b	ab	c	ac	bc	abc	cc	acc	bcc	$abcc$
a	a	e	ab	b	ac	c	abc	bc	acc	cc	$abcc$	bcc
b	b	ab	e	a	bc	abc	c	ac	bcc	$abcc$	cc	acc
ab	ab	b	a	e	abc	bc	ac	c	$abcc$	bcc	acc	cc
c	c	bc	abc	ac	cc	bcc	$abcc$	acc	e	b	ab	a
ac	ac	abc	bc	c	acc	$abcc$	bcc	cc	a	ab	b	e
bc	bc	c	ac	abc	bcc	cc	acc	$abcc$	b	e	a	ab
abc	abc	ac	c	bc	$abcc$	acc	cc	bcc	ab	a	e	b
cc	cc	$abcc$	acc	bcc	e	ab	a	b	c	abc	ac	bc
acc	acc	bcc	cc	$abcc$	a	b	e	ab	ac	bc	c	abc
bcc	bcc	acc	$abcc$	cc	b	a	ab	e	bc	ac	abc	c
$abcc$	$abcc$	cc	bcc	acc	ab	e	b	a	abc	c	bc	ac

Cayley table of the group G defined by (1).

1. The GAP Group, GAP - *groups, Algorithms, and Programming,* Version 4.10.0, 2018, http: //www.gap-system.org

Finding minimal generating sets of finite groups

We can use the algorithm mentioned previously to find a minimal generating set of G as follows :

1. Set $A = \{b\}, v_1 = b, i = 1$.
2. Draw $\mathrm{Cay}\,(G, \{b\})$.

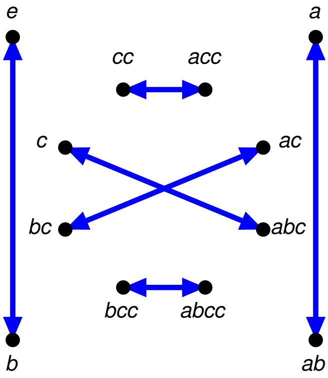

FIGURE – $\overrightarrow{\mathrm{Cay}}_c(G, \{b\})$; blue arcs are induced by b.

Finding minimal generating sets of finite groups

3. Since $\mathrm{Cay}\,(G, \{b\})$ is not connected, set $i = 2$ and $v_2 = ab$.
4. Set $a_2 = b^{-1}(ab) = a$ and $A = \{b, a\}$.
5. Draw $\mathrm{Cay}\,(G, \{b, a\})$.

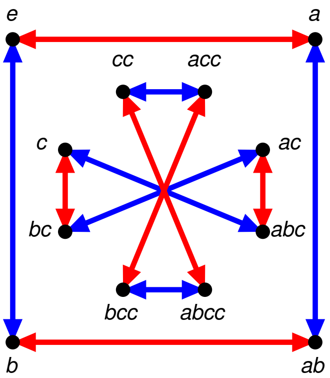

FIGURE – $\overrightarrow{\mathrm{Cay}}_c(G, \{b, a\})$; blue arcs are induced by b and red arcs are induced by a.

Finding minimal generating sets of finite groups

6. Since $\mathrm{Cay}\,(G, \{b, a\})$ is not connected, set $i = 3$ and $v_2 = bc$.
7. Set $a_3 = b^{-1}(bc) = c$ and $A = \{b, a, c\}$.
8. Draw $\mathrm{Cay}\,(G, \{b, a, c\})$.

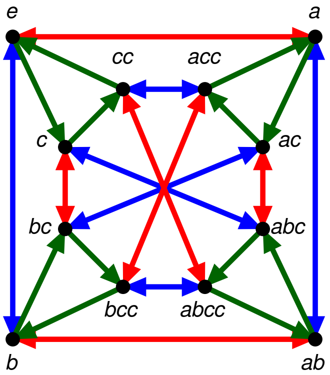

FIGURE – $\overrightarrow{\mathrm{Cay}}_c(G, \{b, a, c\})$; blue arcs are induced by b, red arcs are induced by a, and green arcs are induced by c.

Finding minimal generating sets of finite groups

9. Since $\mathrm{Cay}\,(G, \{b, a, c\})$ is connected and $i = 3$, set $i = 2$.
10. Draw $\mathrm{Cay}\,(G, \{b, c\})$.

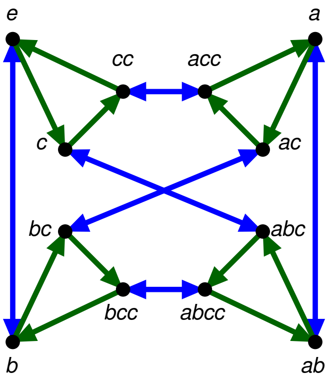

FIGURE – $\overrightarrow{\mathrm{Cay}}_c(G, \{b, c\})$; blue arcs are induced by b and green arcs are induced by c.

Finding minimal generating sets of finite groups

11. Since $\mathrm{Cay}\,(G, \{b, c\})$ is connected, set $A = \{b, c\}$.
12. Since $i = 2$, set $i = 1$.
13. Draw $\mathrm{Cay}\,(G, \{c\})$.

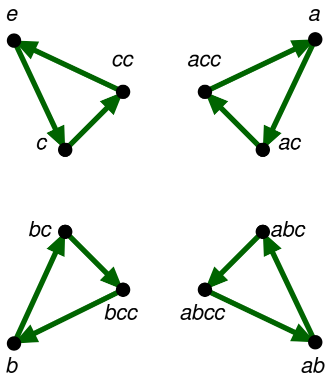

FIGURE – $\overrightarrow{\mathrm{Cay}}_c(G, \{c\})$; green arcs are induced by c.

Finding minimal generating sets of finite groups

14. Since $\mathrm{Cay}\,(G, \{c\})$ is not connected, go to the next step.
15. Since $i = 1$, stop.

This shows that $A = \{b, c\}$ is a minimal generating set of G.

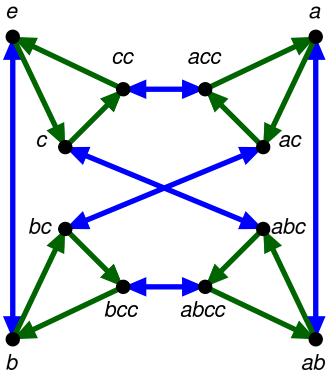

FIGURE – $\overrightarrow{\mathrm{Cay}}_c(G, \{b, c\})$; blue arcs are induced by b and green arcs are induced by c.

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