On the multiplicative order of $\alpha + \alpha^{-1}$ in finite fields of characteristic two

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Introduction

Problem (Blake et al., 1993)

Let $\mathbb{F}_q$ be a finite field with q elements, where q is a power of a prime p.

Let $\mathbb{F}_q^*$ be the multiplicative group of $\mathbb{F}_q$.

If $\alpha \in \mathbb{F}_q^* \backslash \{1\}$ is an element of order $\mathrm{ord}(\alpha)$, can we find $\mathrm{ord}(\alpha + \alpha^{-1})$ from $\mathrm{ord}(\alpha)$?

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Some results

Theorem (Shparlinski, 2001)

If $\gamma \in \mathbb{F}_q^*$ does not belong to any proper subfield of $\mathbb{F}_q$, then at least one of the multiplicative orders of γ and $\gamma + \gamma^{-1}$ exceeds $c(p, \varepsilon)(\ln q)^{4/3-\varepsilon}$, where $c(p, \varepsilon) > 0$ depends only on p and arbitrary $\varepsilon > 0$.

Remark

The theorem above is a particular case of Theorem 4.3 in (von zur Gathen-Shparlinski, 1999).

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Theorem (Shparlinski, 2001)

For any fixed $\varepsilon > 0$ and sufficiently large q, for any positive divisors n and m of $q-1$ with $nm \geq q^{3/2+\varepsilon}$ there exists $\gamma \in \mathbb{F}_q^*$ with

$$\operatorname{ord}(\gamma) = n \quad \text{and} \quad \operatorname{ord}(\gamma + \gamma^{-1}) = m.$$

Dickson polynomials

For each integer $m > 0$ we define the Dickson polynomial of degree m as

$$D_m(x) = \sum_{i=0}^{\lfloor m/2 \rfloor} \frac{m}{m-i} \binom{m-i}{i} (-1)^i x^{m-2i}$$

Properties

- $D_m(x) \in \mathbb{Z}[x]$.
- $D_m(x + x^{-1}) = x^m + x^{-m}$.

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Roots of Dickson polynomials

- Any element $\alpha \in \mathbb{F}_q$ can be written as $\alpha = \gamma + \gamma^{-1}$ for some $\gamma \in \mathbb{F}_{q^2}^*$ (i.e. γ is a root of $x^2 + \alpha x + 1$ in $\mathbb{F}_{q^2}$).
- Finding a root $\alpha \in \mathbb{F}_q$ of $D_m(x)$ amounts to finding some $\gamma \in \mathbb{F}_{q^2}^*$ such that

$$D_m(\gamma + \gamma^{-1}) = \gamma^m + \gamma^{-m} = \gamma^{-m}(\gamma^m + 1)^2 = 0$$

or equivalently

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Roots of Dickson polynomials

A question by Blokhuis et al., 2018

Let m be an integer which divides $q+1 = 2^n + 1$.

Consider the sets

$$S_m := \{\alpha \in \mathbb{F}_q^* : D_m(\alpha) = D_m(\alpha^{-1}) = 0\};$$
$$T_m := \{\alpha \in \mathbb{F}_q^* : D_m(\alpha) = 0, D_m(\alpha^{-1}) \neq 0\}.$$

Are the sets S_m and T_m non-empty?

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The graph associated with the map $x \mapsto x + x^{-1}$

In (Ugolini, 2012) I studied the structure of the graph associated with the map ϑ defined on $\mathbf{P}^1(\mathbb{F}_q) := \mathbb{F}_q \cup \{\infty\}$ as

$$\vartheta : x \mapsto \begin{cases} \infty & \text{if } x \in \{0, \infty\}; \\ x + x^{-1} & \text{otherwise} \end{cases}$$

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A note on the graph's construction

If $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{P}^1(\mathbb{F}_q)$ and $\alpha_2 = \vartheta(\alpha_1), \alpha_3 = \vartheta(\alpha_2)$, then

$$\alpha_1 \longrightarrow \alpha_2 \longrightarrow \alpha_3$$

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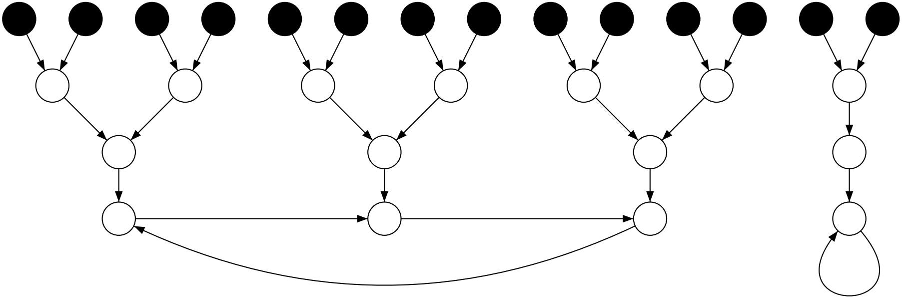
A remark on the elements in S_{q+1} and T_{q+1}

The elements in S_{q+1} and T_{q+1} appear as leaves of the connected components of the graph associated with the map ϑ.

Example: graph associated with ϑ over $\mathbb{F}_q = \mathbb{F}_{2^6}$



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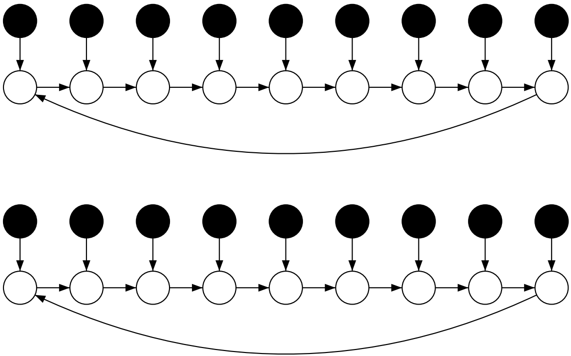

$\boxed{\text{Black nodes: elements in } T_{q+1}}$

About the structure of the graph

The properties of the graph, as

- the depth of the trees (which is the same in any connected component),
- the length of the cycles,

can be explained relating the map ϑ to the duplication map on a certain elliptic curve (a Koblitz curve).

Theorem

If $n := 2^l m$ for some non-negative integer l and some odd integer m, then either all trees in a connected component have depth 1 or $l + 2$.

Example

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- Let γ be an element of $\mathbb{F}_{q^4} \backslash \{0, 1\}$ such that $\mathrm{ord}(\gamma) \mid (q^2 + 1)$.
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- Let γ_i be the i-th iterate of γ, e.g.

$$\begin{aligned}\gamma_1 &= \vartheta(\gamma),\\ \gamma_2 &= \vartheta(\gamma_1) = \vartheta^2(\gamma),\\ \gamma_3 &= \vartheta(\gamma_2) = \vartheta^3(\gamma),\\ &\dots\end{aligned}$$

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Case 1

- ord$(\gamma) \mid (q^2+1)$;
- ord$(\gamma_1) \mid (q+1)$;
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- For example, we can deduce that any element in the cyclic group C_{q+1} of order $q+1$ in $\mathbb{F}_{q^2}^*$ is expressible as

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References I

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