# The intersection problem in $F_n \times \mathbb{Z}^m$

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# Outline

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# Finitely generated free-abelian groups

*This talk is about semidirect products of f.g. free-abelian groups by f.g. free groups (free-abelian-by-free, for short).*

*For all the talk,*

- *elements from $\mathbb{Z}^m$ are row integral vectors of length m, with additive notation;*
- *think endomorphisms of $\mathbb{Z}^m$ as $m \times m$ integral matrices $\mathbf{A}$, acting on the right $\mathbf{u} \mapsto \mathbf{uA}$;*
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# Abelian-by-free groups

- *Let* $X = \{x_1, \ldots, x_n\}$ *be a set, and* $\mathbb{F}_X$ *the free group on it;*
- *fix matrices* $\mathbf{A}_j \in \mathrm{GL}_m(\mathbb{Z})$, $j = 1, \ldots, n$ *and consider the action* $\mathbf{A}_\bullet : \mathbb{F}_X \to \mathrm{Aut}(\mathbb{Z}^m)$, $x_j \mapsto A_j$;
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$$\mathbb{G}_\mathbf{A} = \left\langle \begin{array}{c} x_1, \ldots, x_n \\ t_1, \ldots, t_m \end{array} \;\middle|\; \begin{array}{ll} t_i t_k = t_k t_i & \forall i, k \in [1, m] \\ x_j^{-1} t_i x_j = t^{\mathbf{e}_i \mathbf{A}_j} & \forall i \in [1, m], \forall j \in [1, n] \end{array} \right\rangle$$

- *Notation:* $t^\mathbf{u} := t^{(u_1, \ldots, u_m)} := t_1^{u_1} t_2^{u_2} \cdots t_m^{u_m}$, *where 't' is meaningless;*
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*We have the standard split short exact sequence,*

$$1 \longrightarrow \mathbb{Z}^m \longrightarrow \mathbb{G}_{\mathbf{A}} = \mathbb{F}_X \ltimes_{\mathbf{A}_\bullet} \mathbb{Z}^m \xrightarrow{\pi} \mathbb{F}_X \longrightarrow 1.$$

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*Writting the semidirect relation as $t^{\mathbf{u}} x_j = x_j t^{\mathbf{u}\mathbf{A}_j}$ or $t^{\mathbf{u}} x_j^{-1} = x_j^{-1} t^{\mathbf{u}\mathbf{A}_j^{-1}}$, we get normal forms for elements $g \in \mathbb{G}_{\mathbf{A}}$ as*

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# Subgroups of free-abelian-by-free groups

## Proposition

*Let $\mathbb{G}_{\mathbf{A}} = \mathbb{F}_X \ltimes_{\mathbf{A}_\bullet} \mathbb{Z}^m$. Then, any subgroup $H \leqslant \mathbb{G}_{\mathbf{A}}$ admits the decomposition $H \simeq H\pi \ltimes L_H$, where $H\pi \leqslant \mathbb{F}_X$, $L_H = H \cap \mathbb{Z}^m \leqslant \mathbb{Z}^m$, and the action is the restriction of $\mathbf{A}_\bullet$ to both $H\pi$ and $L_H$, i.e., $H\pi \to \mathrm{Aut}(L_H)$, $w \mapsto \mathbf{A}_{\mathbf{w}|L_H}$.*

## Corollary

*Any subgroup $H$ of a free-abelian-by-free group $\mathbb{G}_{\mathbf{A}}$ is again free-abelian-by-free. Moreover, $H$ is finitely generated if and only if $H\pi$ is finitely generated.*

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*To extend the Stallings bijection*

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*to subgroups of $\mathbb{G}_{\mathbf{A}}$, and use it to solve several algorithmic problems in free-abelian-by-free groups. We shall focus on:*

- *the membership problem;* ✓
- *the subgroup conjugacy problem;* *caution! there are $F_{14} \ltimes \mathbb{Z}^4$ groups with unsolvable CP*
- *the intersection problem;* *caution! $F_2 \times \mathbb{Z}$ is NOT Howson*

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# Outline

1. Free-abelian-by-free groups
2. Stallings' automata
3. Vectored Stallings' automata
4. Membership
5. Intersection

# Stallings' automata

## Definition

*A Stallings automaton over $X$ is a finite $X$-graph $(V, E, q_0)$, such that:*

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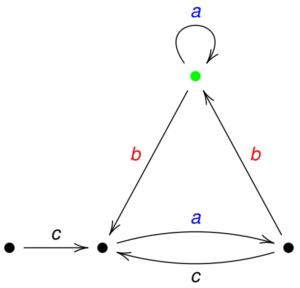


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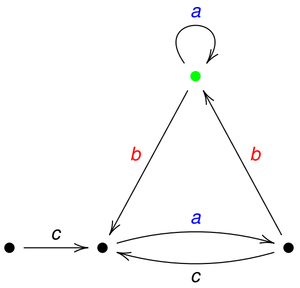


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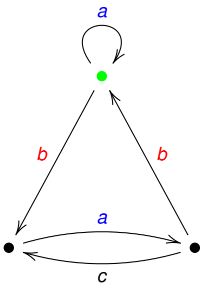

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# Reading the subgroup from the automata

## Definition

*To any given Stallings automaton $\mathcal{A} = (V, E, q_0)$, we associate its* ***language****:*

$$L(\mathcal{A}) = \{ \text{ labels of closed paths at } q_0 \} \leqslant F(A).$$




$$L(\mathcal{A}) = \{1, a, a^{-1}, bab, bc^{-1}b, babab^{-1}cb^{-1}, \ldots\}$$

$$L(\mathcal{A}) \not\ni bc^{-1}bcaa$$

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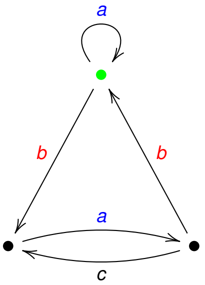
$\mathcal{A}=$ 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# A basis for $L(\mathcal{A})$

## Proposition

*For every Stallings automaton $\mathcal{A} = (V, E, q_0)$, and every maximal tree $T$, the group $L(\mathcal{A})$ is free with free basis*

$$\{x_e = label(T[q_0, \iota e] \cdot e \cdot T[\tau e, q_0]) \in L(\mathcal{A}) \mid e \in EX - ET\},$$

*where $T[p, q]$ denotes the geodesic in $T$ from $p$ to $q$. In particular, $rk(L(\mathcal{A})) = 1 - |V| + |E|$.*

# Constructing the automaton from the subgroup

*Given $H = \langle w_1, \ldots, w_n \rangle \in F(A)$, construct the flower automaton, denoted $\mathcal{F}(H)$.*

*Clearly, $L(\mathcal{F}(H)) = H$.*

*... But $\mathcal{F}(H)$ is not in general deterministic...*

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*If $\mathcal{A} \rightsquigarrow \mathcal{A}'$ is a Stallings folding then $L(\mathcal{A}) = L(\mathcal{A}')$.*

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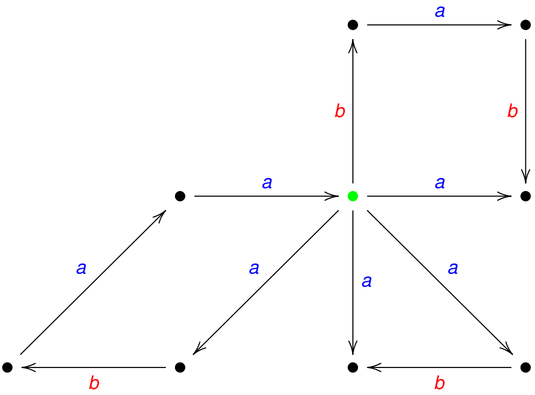


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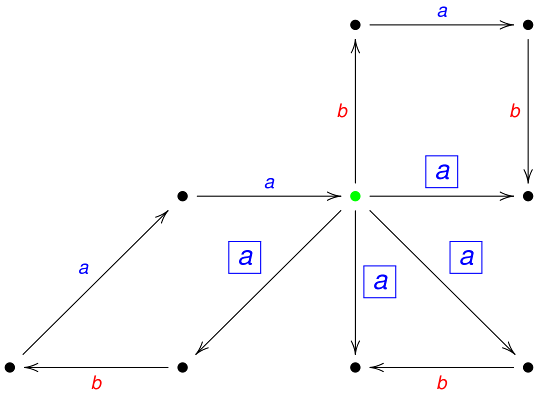


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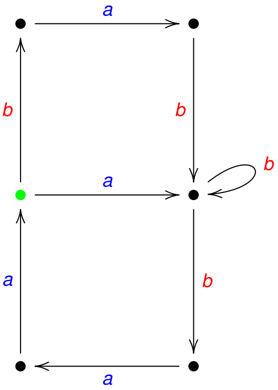


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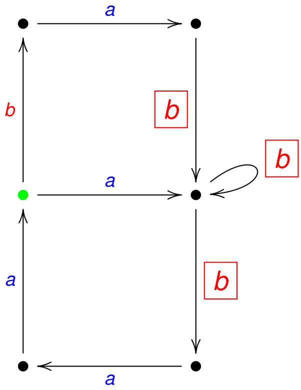


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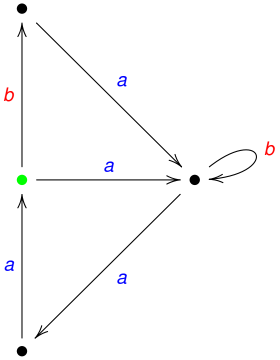


Folding $\#2$.

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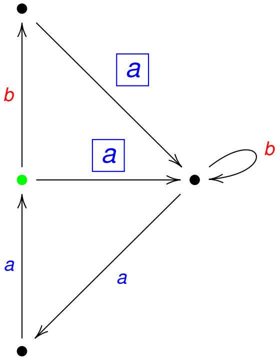


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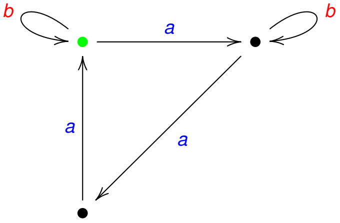


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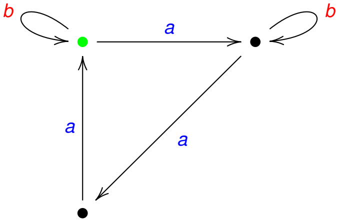


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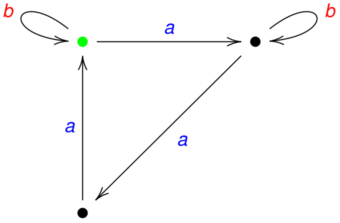


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# Local confluence

It can be shown that

**Proposition**

*The automaton $\Gamma(H)$ does not depend on the sequence of foldings.*

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*The automaton $\Gamma(H)$ does not depend on the generators of $H$.*

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*The following is a well defined bijection:*

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# Outline

# Vectored Stallings' automata

## Definition

*Let us consider now, vectored $X$-automata, i.e., $X$-graphs with vectors assigned at the heads and tails of the edges,*

$$\bullet \xrightarrow{u_1 \quad a \quad u_2} \bullet \,,$$

*reading $t^{-u_1} a t^{u_2} = a t^{u_2 - u_1 A}$ (and the inverse if traversed backwards).*
*... plus a subspace $L \leqslant \mathbb{Z}^m$ attached to the basepoint.*

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*For a f.g. subgroup $H = \langle w_1 t^{u_1}, \ldots, w_r t^{u_r} \rangle$ of $\mathbb{G}_{\mathbf{A}} = \mathbb{F}_n \ltimes_{A_1,\ldots,A_n} \mathbb{Z}^m$, we can also construct a flower automaton.*

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# Abelian moves

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*We need now some extra operations to allow moving abelian mass arround:*

- *edge moves,*
- *vertex moves,*
- *vertex moves at the basepoint,*
- *open foldings,*
- *closed foldings,*
- *increase $L$ to its closure by the labels of all closed paths at •.*

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*A vectored Stallings $A$-automata is a connected and trim vectored $A$-automata satisfying:*
*(i) $\mathcal{A}$ is deterministic,*
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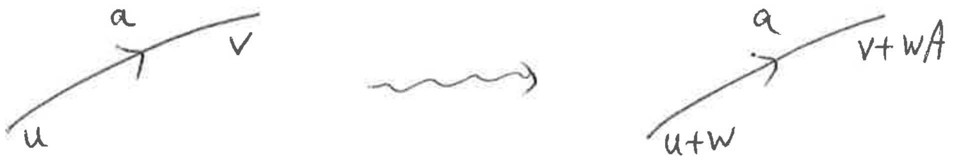
## Edge move




$$\cdots t^{-u} a\, t^{v} \cdots \qquad \cdots t^{-u-w} a\, t^{v+wA}$$

## Vertex move


## vertex move at ⊙

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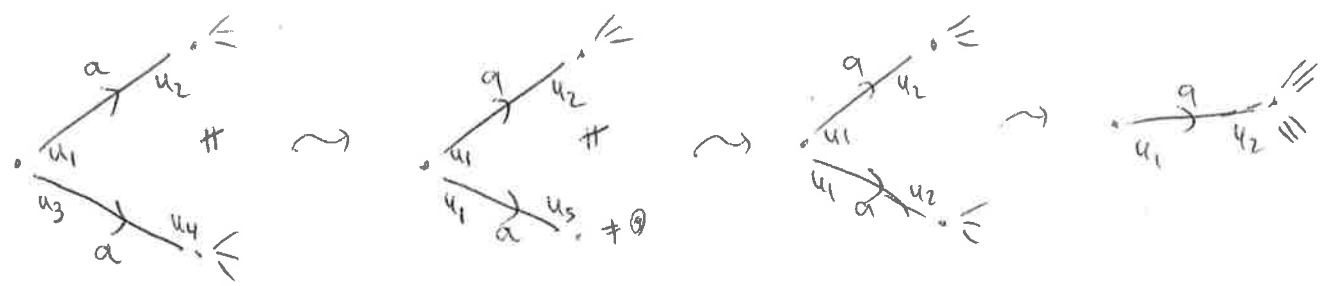
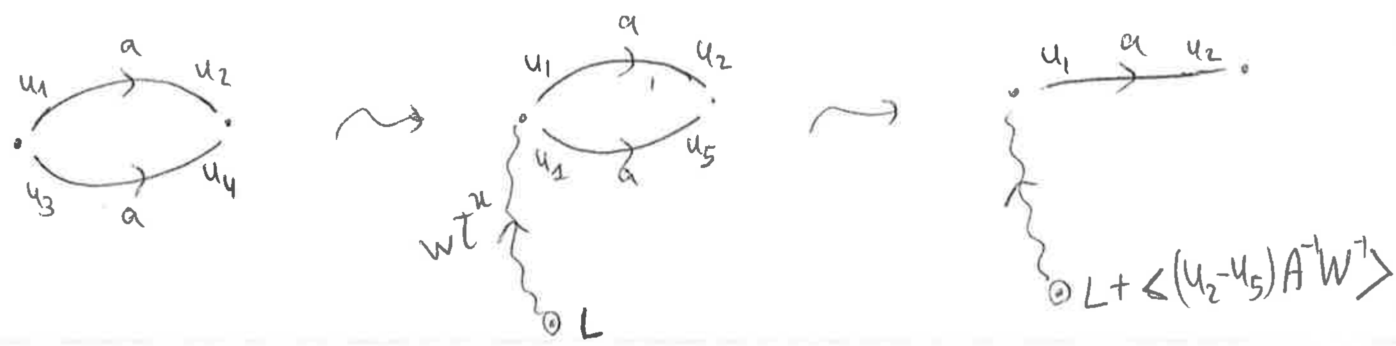



## closed folding




$$w t^u t^{-u_1} a t^{u_2} t^{-u_5} a^{-1} t^{u_1} t^{-u} W^{-1} =$$

$$= w \cancel{a a^{-1}} W^{-1} \cdot t^{\cancel{(u-u_1) W^{-1}} + (u_2-u_5) A^{-1} W^{-1} + \cancel{(u_1-u) W^{-1}}}$$

$$= t^{(u_2-u_5) A^{-1} W^{-1}}$$

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**Lemma**

*With repeated use of the above operations, any vectored $X$-automata $\mathcal{A}$ can be converted into a vectored Stallings $X$-automata $\mathcal{A}'$.*

**Lemma**

*For $H \leqslant \mathbb{G}_{\mathbf{A}}$, the result of folding $\mathcal{F}I(H)$, say $\Gamma(H)$, is uniquely determined by the subgroup $H$, modulo the choice of the maximal tree, and with all vectors around understood 'modulo $L$'.*

**Theorem (Delgado–V., 2016)**

*The following map is a bijection:*

$$\begin{array}{ccc} \{\text{Subgroups of } \mathbb{G}_{\mathbf{A}}\} & \longleftrightarrow & \{\text{Vectored Stallings } X\text{-automata}\} \\ H & \mapsto & \Gamma(H) \\ L(\mathcal{A}) & \leftarrow & \mathcal{A} \end{array}$$

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# Outline

# Membership

**Corollary (Delgado–V., 2016)**

*Membership is solvable in free-abelian-by-free groups.*

**Proof.** Given $w\mathbf{t}^{\mathbf{u}} \in \mathbb{G}_{\mathbf{A}}$ and $H = \langle w_1 \mathbf{t}^{\mathbf{u}_1}, \ldots, w_r \mathbf{t}^{\mathbf{u}_r} \rangle \leqslant \mathbb{G}_{\mathbf{A}}$, do the following:

1- draw the flower automaton $\mathcal{F}l(H)$;

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3- if $w$ is not readable as closed path at •, answer **NO**;

4- otherwise, compute the full label of such a close path, say $w\mathbf{t}^{\mathbf{v}}$;

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# Outline

# The Subgroup Intersection Problem

**(Caution!)**

*The groups $\mathbb{G}_{\mathbf{A}}$ are NOT Howson, in general.*

**Definition (The Subgroup Intersection Problem for $G$)**

***Input:*** $g_1, \ldots, g_r, g_1', \ldots, g_s' \in G$
***Decide:*** $\langle g_1, \ldots, g_r \rangle \cap \langle g_1', \ldots, g_s' \rangle$ *is f.g. and, if so, compute generators.*

**Theorem (Delgado–V., 2017)**

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*The groups $\mathbb{G}_{\mathbf{A}}$ are NOT Howson, in general.*

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*Given $H, K \leqslant_{fg} \mathbb{G}_{\mathbf{A}}$, we know that*

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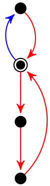
$H$ 

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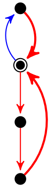

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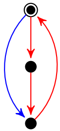

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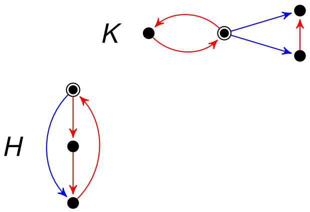

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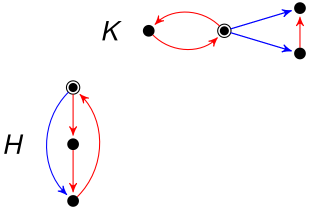

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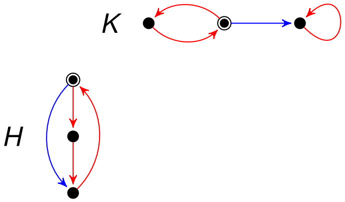

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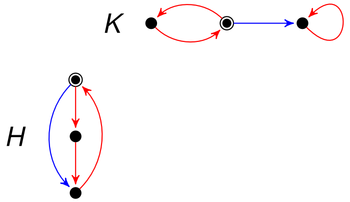

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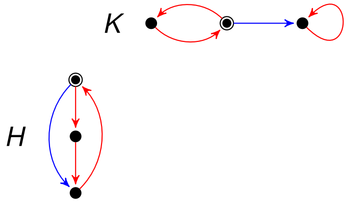

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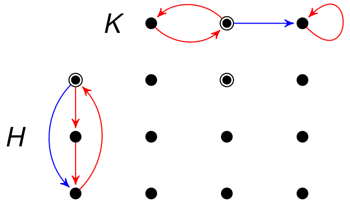

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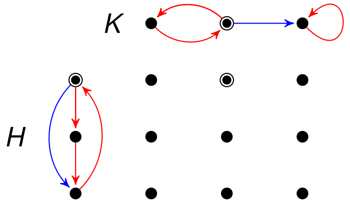

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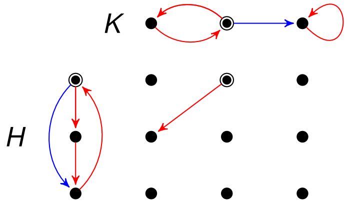

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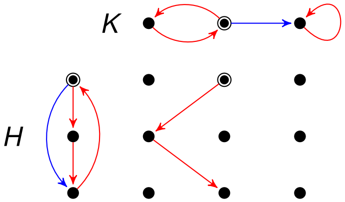

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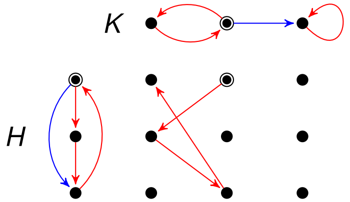

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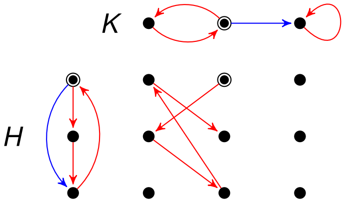

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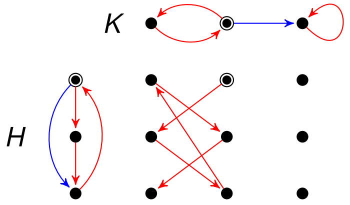

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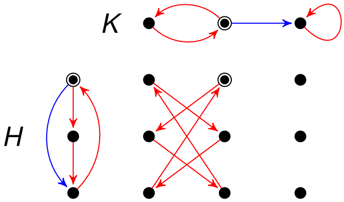

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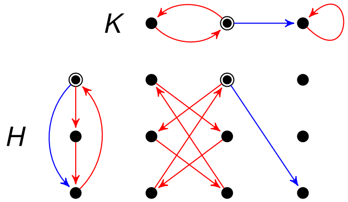

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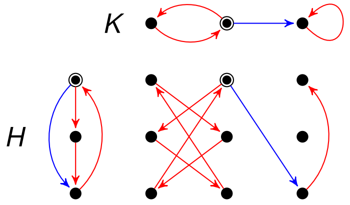

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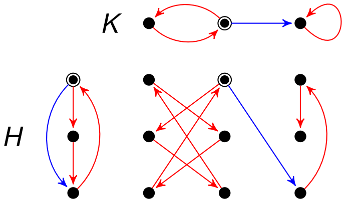

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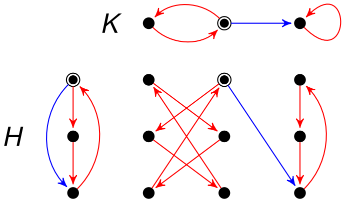

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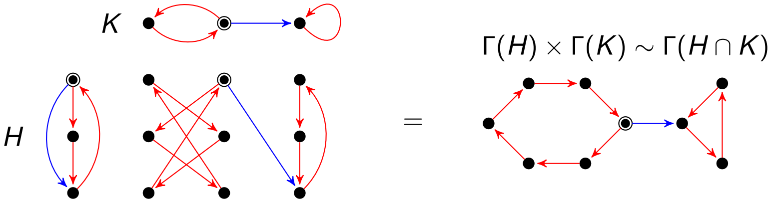

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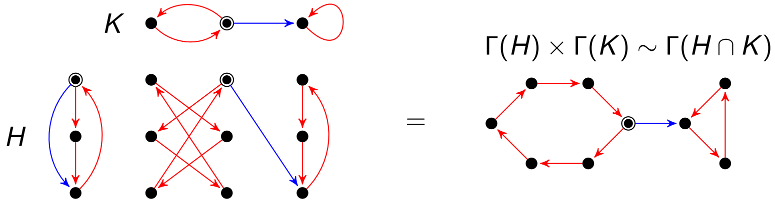


Therefore, $H \cap K = \langle x^6, yx^3y^{-1} \rangle$.

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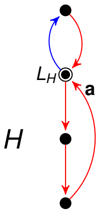

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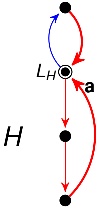
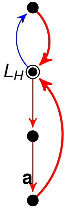


# Intersections in $\mathbb{F}_n \times \mathbb{Z}^m$

Let $H = \langle t^{L_H}, x^3\, t^{\mathbf{a}}, yx \rangle$, $K = \langle t^{L_K}, x^2\, t^{\mathbf{d}}, yxy^{-1} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$

$H$



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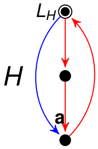

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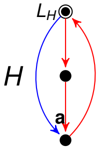

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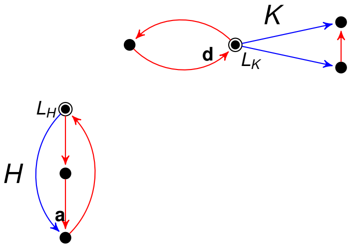

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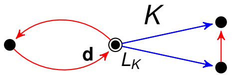


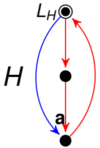

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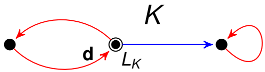


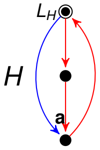

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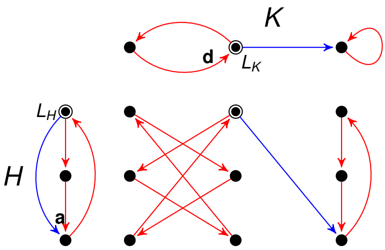

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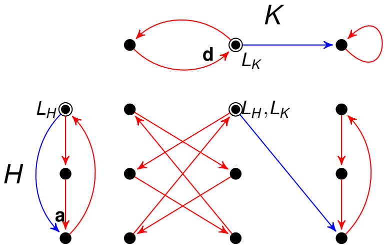

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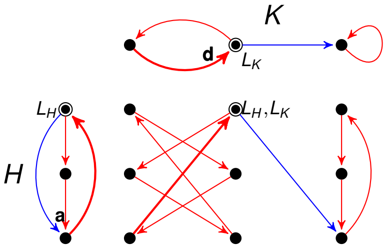

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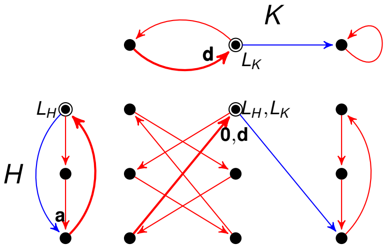

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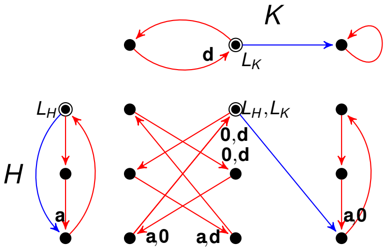

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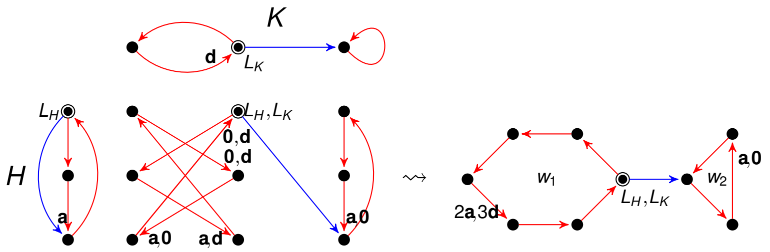

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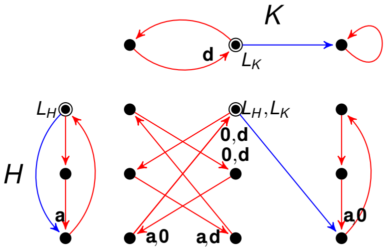


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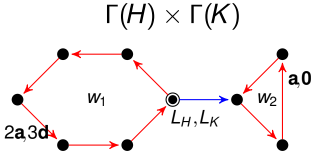

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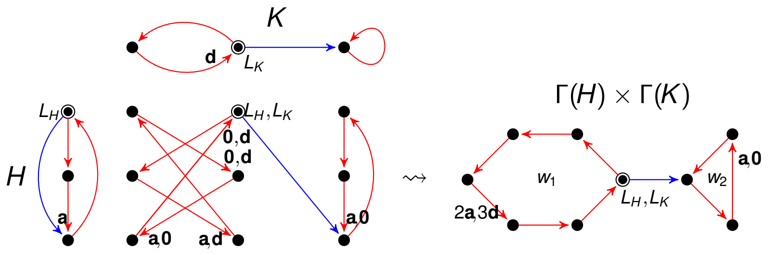

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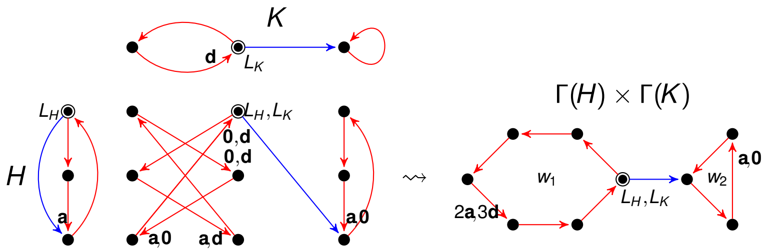


**Claim:**

$$H \cap K = \langle u\, t^{\mathbf{a}} \mid u\, t^{\mathbf{a}} \text{ is componentwise-readable in } \Gamma(H) \times \Gamma(K) \rangle$$

$$\rangle$$

# Intersections in $\mathbb{F}_n \times \mathbb{Z}^m$

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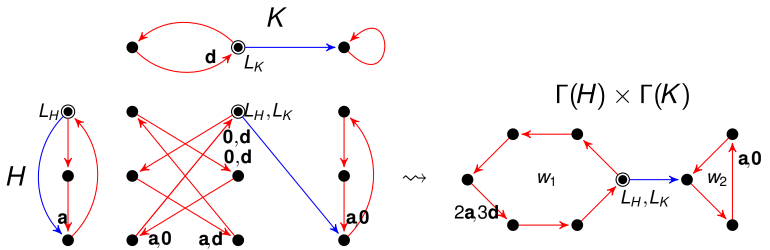


**Claim:**

$$H \cap K \;=\; \langle u\, t^{\mathbf{a}} \mid u\, t^{\mathbf{a}} \text{ is componentwise-readable in } \Gamma(H) \times \Gamma(K) \rangle$$

$$(H \cap K)\pi \;=\; \Big\langle w \in \mathbb{F}_{\{w_1, w_2\}} \mid w(w_1 t^{2\mathbf{a}}, w_2 t^{\mathbf{a}})\, t^{L_H} \,\cap\, w(w_1 t^{3\mathbf{d}}, w_2 t^{\mathbf{0}})\, t^{L_K} \;\neq\; \varnothing \Big\rangle$$

$$\Big\rangle$$

# Intersections in $\mathbb{F}_n \times \mathbb{Z}^m$

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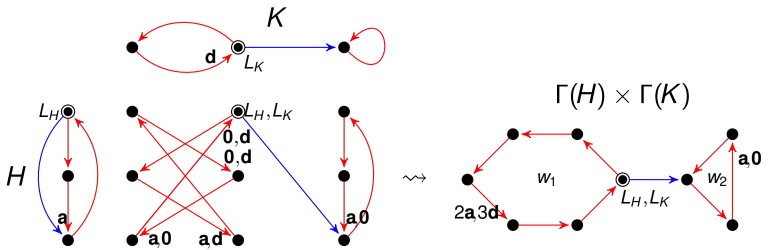


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$$= \langle w \in \mathbb{F}_{\{w_1, w_2\}} \mid w^{\mathrm{ab}} \begin{bmatrix} 2\mathbf{a} - 3\mathbf{d} \\ \mathbf{a} - \mathbf{0} \end{bmatrix} \in L_H + L_K \rangle$$

# Intersections in $\mathbb{F}_n \times \mathbb{Z}^m$

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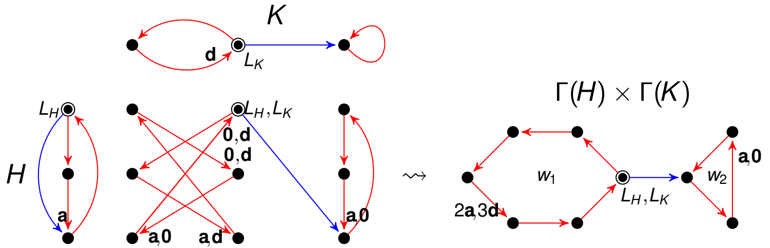


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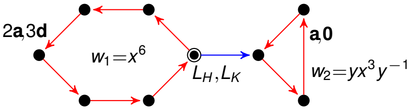
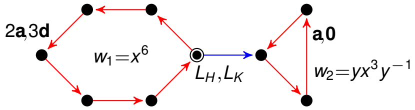
$$= (L_H + L_K)\mathbf{B}^{-1}\rho^{-1}, \text{ where } \mathbf{B} = \begin{bmatrix} 2\mathbf{a} - 3\mathbf{d} \\ \mathbf{a} - \mathbf{0} \end{bmatrix} \text{ and } \rho = \mathrm{ab}\,.$$

# Intersections in $\mathbb{F}_n \times \mathbb{Z}^m$



# Intersections in $\mathbb{F}_n \times \mathbb{Z}^m$

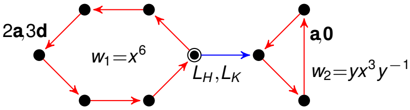



$$\mathbf{B} = \begin{bmatrix} 2\mathbf{a} - 3\mathbf{d} \\ \mathbf{a} - \mathbf{0} \end{bmatrix}$$

$$M = (L_H + L_K)\mathbf{B}^{-1}$$

We have that $(H \cap K)\pi = (L_H + L_K)\mathbf{B}^{-1}\rho^{-1} = M\rho^{-1}$, i.e.,

# Intersections in $\mathbb{F}_n \times \mathbb{Z}^m$




$$\mathbf{B} = \begin{bmatrix} \mathbf{2a}-\mathbf{3d} \\ \mathbf{a}-\mathbf{0} \end{bmatrix}$$

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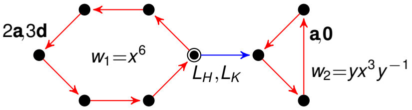
We have that $(H \cap K)\pi = (L_H + L_K)\mathbf{B}^{-1}\rho^{-1} = M\rho^{-1}$, i.e.,

$$\begin{array}{ccccccccc} \mathbb{F}_2 \geqslant & H\pi \cap K\pi & = & \mathbb{F}_{\{w_1, w_2\}} & \xrightarrow{\rho} & \mathbb{Z}^2 & \xrightarrow{\mathbf{B}} & \mathbb{Z}^2 \\ & \triangledown & & \triangledown & & \triangledown & & \triangledown \\ 1 \neq & (H \cap K)\pi & = & M\rho^{-1} & \longleftarrow & M & \longleftarrow & L_H + L_K \end{array}$$

# Intersections in $\mathbb{F}_n \times \mathbb{Z}^m$

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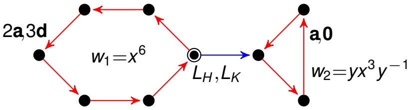
Then, $\Gamma((H \cap K)\pi, \{w_1, w_2\}) = \Gamma(M\rho^{-1}, \{w_1, w_2\})$

# Intersections in $\mathbb{F}_n \times \mathbb{Z}^m$




$$\mathbf{B} = \begin{bmatrix} 2\mathbf{a} - 3\mathbf{d} \\ \mathbf{a} - \mathbf{0} \end{bmatrix}$$

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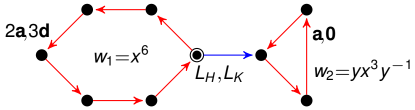
\end{array}$$

$$\begin{aligned}
\text{Then,}\ \ \Gamma((H \cap K)\pi, \{w_1, w_2\}) &= \Gamma(M\rho^{-1}, \{w_1, w_2\}) \\
&= Sch\left(M\rho^{-1}, \{w_1, w_2\}\right)
\end{aligned}$$

# Intersections in $\mathbb{F}_n \times \mathbb{Z}^m$




$$\mathbf{B} = \begin{bmatrix} 2\mathbf{a} - 3\mathbf{d} \\ \mathbf{a} - \mathbf{0} \end{bmatrix}$$

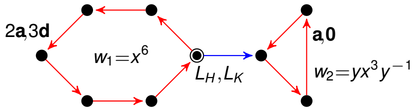
$$M = (L_H + L_K)\mathbf{B}^{-1}$$

We have that $(H \cap K)\pi = (L_H + L_K)\mathbf{B}^{-1}\rho^{-1} = M\rho^{-1}$, i.e.,

$$\begin{tikzcd}
\mathbb{F}_2 \geqslant & H\pi \cap K\pi \arrow[d, phantom, "\triangledown"] & = & \mathbb{F}_{\{w_1, w_2\}} \arrow[r, two heads, "\rho"] \arrow[d, phantom, "\triangledown"] & \mathbb{Z}^2 \arrow[r, "\mathbf{B}"] \arrow[d, phantom, "\triangledown"] & \mathbb{Z}^2 \arrow[d, phantom, "\triangledown"] \\
& 1 \neq (H \cap K)\pi & = & M\rho^{-1} & M \arrow[l, mapsto] & L_H + L_K \arrow[l, mapsto]
\end{tikzcd}$$

$$\begin{aligned}
\text{Then, } \Gamma((H \cap K)\pi, \{w_1, w_2\}) &= \Gamma(M\rho^{-1}, \{w_1, w_2\}) \\
&= Sch\left(M\rho^{-1}, \{w_1, w_2\}\right) \\
&= Cay\left(\mathbb{F}_{\{w_1, w_2\}}/M\rho^{-1}, \{[w_1], [w_2]\}\right)
\end{aligned}$$

# Intersections in $\mathbb{F}_n \times \mathbb{Z}^m$




$$\mathbf{B} = \begin{bmatrix} \mathbf{2a} & \mathbf{3d} \\ \mathbf{a} & \mathbf{0} \end{bmatrix}$$

$$M = (L_H + L_K)\mathbf{B}^{-1}$$

We have that $(H \cap K)\pi = (L_H + L_K)\mathbf{B}^{-1}\rho^{-1} = M\rho^{-1}$, i.e.,

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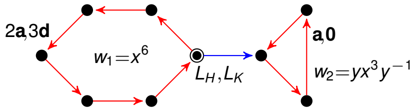
\end{array}$$

Then,
$$\begin{aligned}
\Gamma((H \cap K)\pi, \{w_1, w_2\}) &= \Gamma(M\rho^{-1}, \{w_1, w_2\}) \\
&= \mathit{Sch}\left(M\rho^{-1}, \{w_1, w_2\}\right) \\
&= \mathit{Cay}\left(\mathbb{F}_{\{w_1, w_2\}}/M\rho^{-1}, \{[w_1], [w_2]\}\right) \\
&= \mathit{Cay}\left(\mathbb{Z}^2/M, \{\mathbf{e}_1, \mathbf{e}_2\}\right)
\end{aligned}$$

# Intersections in $\mathbb{F}_n \times \mathbb{Z}^m$




$$\mathbf{B} = \begin{bmatrix} 2\mathbf{a} - 3\mathbf{d} \\ \mathbf{a} - \mathbf{0} \end{bmatrix}$$

$$\langle \mathbf{M} \rangle = M = (L_H + L_K)\mathbf{B}^{-1}$$

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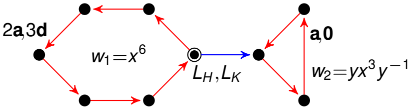
&= Sch\left(M\rho^{-1}, \{w_1, w_2\}\right) \\
&= Cay\left(\mathbb{F}_{\{w_1, w_2\}}/M\rho^{-1}, \{[w_1], [w_2]\}\right) \\
&= Cay\left(\mathbb{Z}^2/M, \{\mathbf{e}_1, \mathbf{e}_2\}\right)
\end{aligned}$$

# Intersections in $\mathbb{F}_n \times \mathbb{Z}^m$




$$\mathbf{B} = \begin{bmatrix} \mathbf{2a}-\mathbf{3d} \\ \mathbf{a}-\mathbf{0} \end{bmatrix}$$

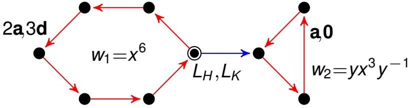
$$\langle \mathbf{M} \rangle = M = (L_H + L_K)\mathbf{B}^{-1}$$

We have that $(H \cap K)\pi = (L_H + L_K)\mathbf{B}^{-1}\rho^{-1} = M\rho^{-1}$, i.e.,

$$\begin{array}{ccccccccc} \mathbb{F}_2 \geqslant & H\pi \cap K\pi & = & \mathbb{F}_{\{w_1, w_2\}} & \xrightarrow{\rho} & \mathbb{Z}^2 & \xrightarrow{\mathbf{B}} & \mathbb{Z}^2 \\ & \triangledown & & \triangledown & & \triangledown & & \triangledown \\ 1 \neq & (H \cap K)\pi & = & M\rho^{-1} & \longleftarrow & M & \longleftarrow & L_H + L_K \end{array}$$

$$\begin{aligned} \text{Then, } \Gamma((H \cap K)\pi, \{w_1, w_2\}) &= \Gamma(M\rho^{-1}, \{w_1, w_2\}) \\ &= Sch\left(M\rho^{-1}, \{w_1, w_2\}\right) \\ &= Cay\left(\mathbb{F}_{\{w_1, w_2\}}/M\rho^{-1}, \{[w_1], [w_2]\}\right) \\ &= Cay\left(\mathbb{Z}^2/\operatorname{row}(\mathbf{M}), \{\mathbf{e}_1, \mathbf{e}_2\}\right) \end{aligned}$$

# Intersections in $\mathbb{F}_n \times \mathbb{Z}^m$




$$\mathbf{B} = \begin{bmatrix} 2\mathbf{a} - 3\mathbf{d} \\ \mathbf{a} - \mathbf{0} \end{bmatrix}$$

$$\langle \mathbf{M} \rangle = M = (L_H + L_K)\mathbf{B}^{-1}$$

$$\mathbf{PMQ} = \mathbf{D} = \operatorname{diag}(\delta_1, \delta_2)$$

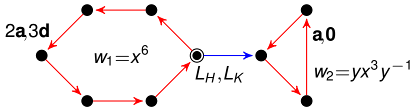
We have that $(H \cap K)\pi = (L_H + L_K)\mathbf{B}^{-1}\rho^{-1} = M\rho^{-1}$, i.e.,

$$\begin{array}{ccccccccc}
\mathbb{F}_2 \geqslant & H\pi \cap K\pi & = & \mathbb{F}_{\{w_1, w_2\}} & \overset{\rho}{\twoheadrightarrow} & \mathbb{Z}^2 & \xrightarrow{\mathbf{B}} & \mathbb{Z}^2 \\
& \triangledown & & \triangledown & & \triangledown & & \triangledown \\
1 \neq & (H \cap K)\pi & = & M\rho^{-1} & \longleftarrow & M & \longleftarrow & L_H + L_K
\end{array}$$

Then, $\Gamma((H \cap K)\pi, \{w_1, w_2\}) = \Gamma(M\rho^{-1}, \{w_1, w_2\})$

$$\begin{aligned}
&= Sch\left(M\rho^{-1}, \{w_1, w_2\}\right) \\
&= Cay\left(\mathbb{F}_{\{w_1, w_2\}}/M\rho^{-1}, \{[w_1], [w_2]\}\right) \\
&= Cay\left(\mathbb{Z}^2/\operatorname{row}(\mathbf{M}), \{\mathbf{e}_1, \mathbf{e}_2\}\right) \\
&= Cay\left(\mathbb{Z}^2/\operatorname{row}(\mathbf{D}), \{\mathbf{e}_1\mathbf{Q}, \mathbf{e}_2\mathbf{Q}\}\right)
\end{aligned}$$

# Intersections in $\mathbb{F}_n \times \mathbb{Z}^m$




$$\mathbf{B} = \begin{bmatrix} \mathbf{2a}-\mathbf{3d} \\ \mathbf{a}-\mathbf{0} \end{bmatrix}$$

$$\langle \mathbf{M} \rangle = M = (L_H + L_K)\mathbf{B}^{-1}$$

$$\mathbf{PMQ} = \mathbf{D} = \operatorname{diag}(\delta_1, \delta_2)$$

We have that $(H \cap K)\pi = (L_H + L_K)\mathbf{B}^{-1}\rho^{-1} = M\rho^{-1}$, i.e.,

$$\begin{array}{ccccccccc} \mathbb{F}_2 \geqslant & H\pi \cap K\pi & = & \mathbb{F}_{\{w_1, w_2\}} & \overset{\rho}{\twoheadrightarrow} & \mathbb{Z}^2 & \overset{\mathbf{B}}{\longrightarrow} & \mathbb{Z}^2 \\ & \triangledown & & \triangledown & & \triangledown & & \triangledown \\ & 1 \neq (H \cap K)\pi & = & M\rho^{-1} & \longleftarrow\!\shortmid & M & \longleftarrow\!\shortmid & L_H + L_K \end{array}$$

Then,

$$\begin{aligned} \Gamma((H \cap K)\pi, \{w_1, w_2\}) &= \Gamma(M\rho^{-1}, \{w_1, w_2\}) \\ &= Sch\left(M\rho^{-1}, \{w_1, w_2\}\right) \\ &= Cay\left(\mathbb{F}_{\{w_1, w_2\}}/M\rho^{-1}, \{[w_1], [w_2]\}\right) \\ &= Cay\left(\mathbb{Z}^2/\operatorname{row}(\mathbf{M}), \{\mathbf{e}_1, \mathbf{e}_2\}\right) \\ &= Cay\left(\mathbb{Z}^2/\operatorname{row}(\mathbf{D}), \{\mathbf{e}_1\mathbf{Q}, \mathbf{e}_2\mathbf{Q}\}\right) \\ &= Cay(\mathbb{Z}/\delta_1\mathbb{Z} \oplus \mathbb{Z}/\delta_2\mathbb{Z}, \{\mathbf{e}_1\mathbf{Q}, \mathbf{e}_2\mathbf{Q}\}). \end{aligned}$$

# Intersection example: case 1

$$H = \langle t^{L_H}, x^3\, t^{\mathbf{a}}, yx \rangle,\ K = \langle t^{L_K}, x^2\, t^{\mathbf{d}}, yxy^{-1} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$$

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**Case 1:** $\mathbf{a} = (1,0)$, $\mathbf{d} = (0,1)$, $L_H = \langle (0,6) \rangle$, $L_K = \langle (3,-3) \rangle$.

# Intersection example: case 1

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Then, $\mathbf{B} = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}$, $\mathbf{M} = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$.

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$\Gamma((H \cap K)\pi, \{w_1, w_2\}) = Cay\left(\mathbb{Z}/1\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}\,,\ \{(1, -1), (0, 1)\}\right)$

# Intersection example: case 1

$$H = \langle t^{L_H}, x^3\, t^{\mathbf{a}}, yx\rangle,\ K = \langle t^{L_K}, x^2\, t^{\mathbf{d}}, yxy^{-1}\rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$$

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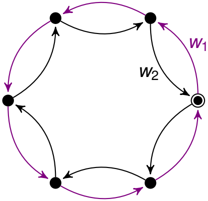

# Intersection example: case 1

$$H = \langle t^{L_H}, x^3\, t^{\mathbf{a}}, yx \rangle,\ K = \langle t^{L_K}, x^2\, t^{\mathbf{d}}, yxy^{-1} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$$

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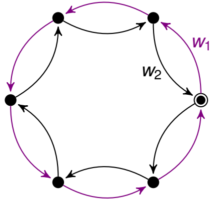
Then, $\mathbf{B} = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}$, $\mathbf{M} = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$. After replacing $w_1 \to x^6 t^{(2,0),(0,3)}$, $w_2 \to yx^3y^{-1} t^{(1,0),(0,0)}$ and folding:



# Intersection example: case 1

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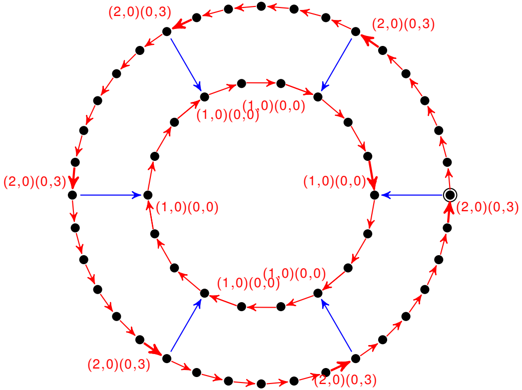

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$$H = \langle t^{L_H}, x^3\, t^{\mathbf{a}}, yx\rangle,\ K = \langle t^{L_K}, x^2\, t^{\mathbf{d}}, yxy^{-1}\rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$$

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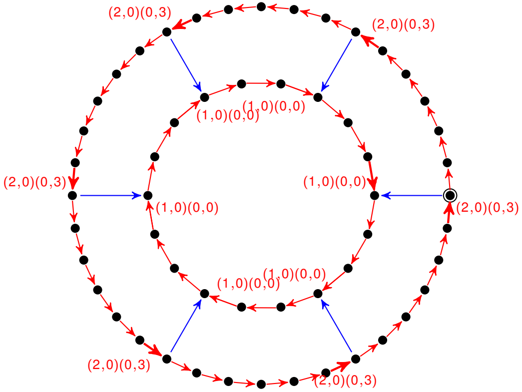
Then, $\mathbf{B} = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}$, $\mathbf{M} = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$. After normalizing w.r.t. an spanning tree:



# Intersection example: case 1

$$H = \langle t^{L_H}, x^3\, t^{\mathbf{a}}, yx \rangle,\ K = \langle t^{L_K}, x^2\, t^{\mathbf{d}}, yxy^{-1} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$$

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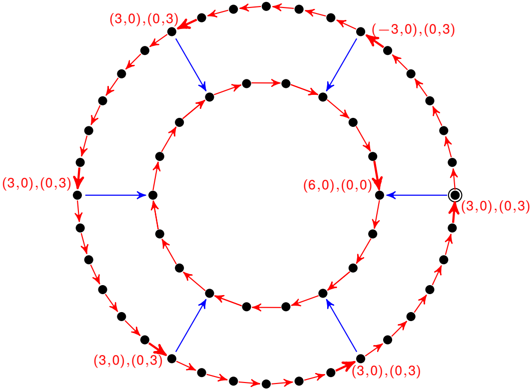

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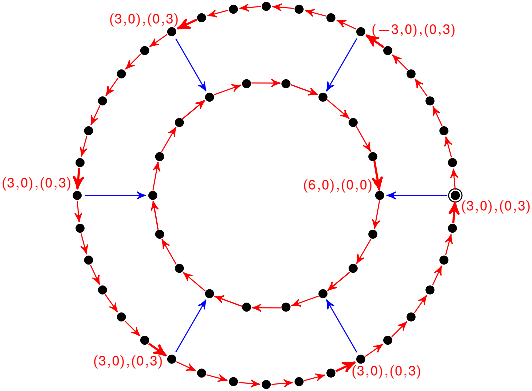
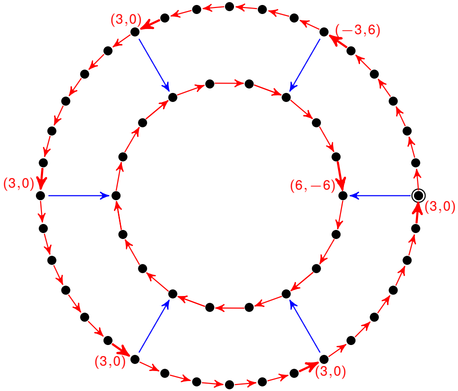


# Intersection example: case 1

$$H = \langle t^{L_H}, x^3\, t^{\mathbf{a}}, yx \rangle,\ K = \langle t^{L_K}, x^2\, t^{\mathbf{d}}, yxy^{-1} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$$

**Case 1:** $\mathbf{a} = (1,0)$, $\mathbf{d} = (0,1)$, $L_H = \langle (0,6) \rangle$, $L_K = \langle (3,-3) \rangle$.

Then, $\mathbf{B} = \left[\begin{smallmatrix} 2 & -3 \\ 1 & 0 \end{smallmatrix}\right]$, $\mathbf{M} = \left[\begin{smallmatrix} -2 & 4 \\ 1 & 1 \end{smallmatrix}\right]$, $\mathbf{Q} = \left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]$, $\mathbf{D} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 6 \end{smallmatrix}\right]$. Finally, after equalizing the abelian labels we obtain $\Gamma(H \cap K)$:



# Intersection example: case 1

Therefore, $H = \langle t^{(0,6)}, x^3\, t^{(1,0)}, yx \rangle$ $K = \langle t^{(3,-3)}, x^2\, t^{(0,1)}, yxy^{-1} \rangle$ and

$$H \cap K = \left\langle \begin{array}{cc} x^6 yx^3y^{-1}\, t^{(3,0)}, & x^{30} yx^3y^{-1} x^{-24}\, t^{(3,0)}, \\ x^{12} yx^3y^{-1} x^{-6}\, t^{(3,0)}, & x^{36}\, t^{(12,6)}, \\ x^{18} yx^3y^{-1} x^{-12}\, t^{(3,0)}, & yx^{18}y^{-1}\, t^{(6,-6)}, \\ x^{24} yx^3y^{-1} x^{-18}\, t^{(3,0)} & \end{array} \right\rangle$$

Note, for example, that

$$H \cap K \ni x^{36} t^{(12,6)} = \begin{cases} = \left(x^3 t^{(1,0)}\right)^{12} t^{(0,6)} \in H, \\ = \left(x^2 t^{(0,1)}\right)^{18} \left(t^{(3,-3)}\right)^4 \in K. \end{cases}$$

And that $x^6 \in H\pi \cap K\pi$ but $x^6 \notin (H \cap K)\pi$ since

$$x^6 t^{(2,0)+\lambda(0,6)} \in H$$

$$x^6 t^{(0,3)+\mu(3,-3)} \in K$$

but $\Big((2,0) + \langle (0,6) \rangle\Big) \cap \Big((0,3) + \langle (3,-3) \rangle\Big) = \emptyset.$

# Intersection example: case 2

$$H = \langle t^{L_H}, x^3\, t^{\mathbf{a}}, yx \rangle,\ K = \langle t^{L_K}, x^2\, t^{\mathbf{d}}, yxy^{-1} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$$

# Intersection example: case 2

$$H = \langle t^{L_H}, x^3\, t^{\mathbf{a}}, yx \rangle,\ K = \langle t^{L_K}, x^2\, t^{\mathbf{d}}, yxy^{-1} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$$

**Case 2:** $\mathbf{a} = (3,3)$, $\mathbf{d} = (2,2)$, $L_H = \langle (1,2) \rangle$, $L_K = \langle (0,0) \rangle$.

# Intersection example: case 2

$$H = \langle t^{L_H}, x^3\, t^{\mathbf{a}}, yx \rangle,\ K = \langle t^{L_K}, x^2\, t^{\mathbf{d}}, yxy^{-1} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$$

**Case 2:** $\mathbf{a} = (3,3)$, $\mathbf{d} = (2,2)$, $L_H = \langle (1,2) \rangle$, $L_K = \langle (0,0) \rangle$.

Then, $\delta_1 = 1$, $\delta_2 = 0$ and so, $\Gamma(H \cap K) = Cay\,(\mathbb{Z}, \{0,1\})$.

# Intersection example: case 2

$$H = \langle t^{L_H}, x^3\, t^{\mathbf{a}}, yx \rangle,\ K = \langle t^{L_K}, x^2\, t^{\mathbf{d}}, yxy^{-1} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$$

**Case 2:** $\mathbf{a} = (3,3)$, $\mathbf{d} = (2,2)$, $L_H = \langle (1,2) \rangle$, $L_K = \langle (0,0) \rangle$.

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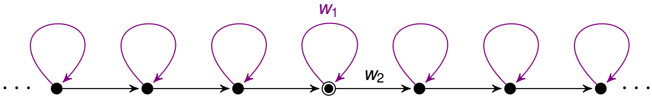

# Intersection example: case 2

$$H = \langle t^{L_H}, x^3\, t^{\mathbf{a}}, yx \rangle,\ K = \langle t^{L_K}, x^2\, t^{\mathbf{d}}, yxy^{-1} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$$

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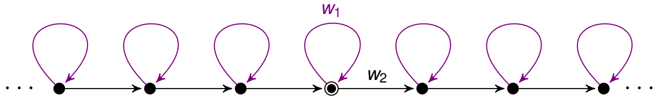


After replacing, folding, normalizing, and equalizing, we obtain $\Gamma(H \cap K)$:

# Intersection example: case 2

$$H = \langle t^{L_H}, x^3\, t^{\mathbf{a}}, yx \rangle,\ K = \langle t^{L_K}, x^2\, t^{\mathbf{d}}, yxy^{-1} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$$

**Case 2:** $\mathbf{a} = (3,3)$, $\mathbf{d} = (2,2)$, $L_H = \langle (1,2) \rangle$, $L_K = \langle (0,0) \rangle$.
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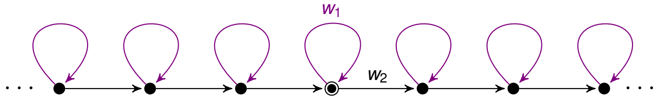
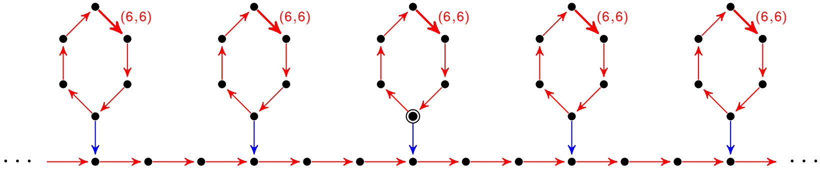



After replacing, folding, normalizing, and equalizing, we obtain $\Gamma(H \cap K)$:



# Intersection example: case 3

$$H = \langle t^{L_H}, x^3\, t^{\mathbf{a}}, yx \rangle,\ K = \langle t^{L_K}, x^2\, t^{\mathbf{d}}, yxy^{-1} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$$

# Intersection example: case 3

$$H = \langle t^{L_H}, x^3\, t^{\mathbf{a}}, yx \rangle,\ K = \langle t^{L_K}, x^2\, t^{\mathbf{d}}, yxy^{-1} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$$

**Case 3:** $\mathbf{a} = (3,3)$, $\mathbf{d} = (2,2)$, $L_H = \langle (2,2) \rangle$, $L_K = \langle (0,0) \rangle$.

# Intersection example: case 3

$$H = \langle t^{L_H}, x^3\, t^{\mathbf{a}}, yx \rangle,\ K = \langle t^{L_K}, x^2\, t^{\mathbf{d}}, yxy^{-1} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$$

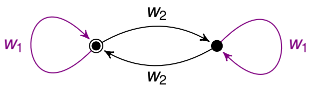
**Case 3:** $\mathbf{a} = (3,3)$, $\mathbf{d} = (2,2)$, $L_H = \langle (2,2) \rangle$, $L_K = \langle (0,0) \rangle$.

Then, $\delta_1 = 1$, $\delta_2 = 2$ and so, $\Gamma(H \cap K) = \mathit{Cay}\,(\mathbb{Z}/2\mathbb{Z}\,, \{0,1\})$.

# Intersection example: case 3

$$H = \langle t^{L_H}, x^3\, t^{\mathbf{a}}, yx \rangle,\ K = \langle t^{L_K}, x^2\, t^{\mathbf{d}}, yxy^{-1} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$$

**Case 3:** $\mathbf{a} = (3,3)$, $\mathbf{d} = (2,2)$, $L_H = \langle (2,2) \rangle$, $L_K = \langle (0,0) \rangle$.
Then, $\delta_1 = 1$, $\delta_2 = 2$ and so, $\Gamma(H \cap K) = \mathit{Cay}\,(\mathbb{Z}/2\mathbb{Z}, \{0,1\})$.



# Intersection example: case 3

$$H = \langle t^{L_H}, x^3\, t^{\mathbf{a}}, yx \rangle,\ K = \langle t^{L_K}, x^2\, t^{\mathbf{d}}, yxy^{-1} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$$

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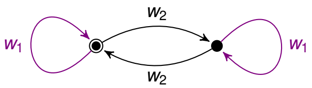
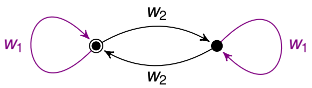



After replacing, folding, normalizing, and equalizing, we obtain $\Gamma(H \cap K)$:

# Intersection example: case 3

$$H = \langle t^{L_H}, x^3\, t^{\mathbf{a}}, yx \rangle,\ K = \langle t^{L_K}, x^2\, t^{\mathbf{d}}, yxy^{-1} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$$

**Case 3:** $\mathbf{a} = (3,3)$, $\mathbf{d} = (2,2)$, $L_H = \langle (2,2) \rangle$, $L_K = \langle (0,0) \rangle$.
Then, $\delta_1 = 1$, $\delta_2 = 2$ and so, $\Gamma(H \cap K) = Cay\,(\mathbb{Z}/2\mathbb{Z}\,, \{0,1\})$.

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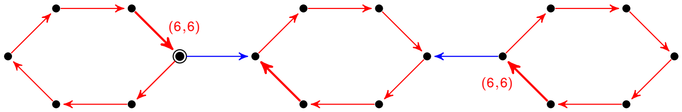

# Intersection example: case 4

$$H = \langle t^{L_H}, x^3\, t^{\mathbf{a}}, yx \rangle,\ K = \langle t^{L_K}, x^2\, t^{\mathbf{d}}, yxy^{-1} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$$

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**Case 4:** $\mathbf{a} = (6,6)$, $\mathbf{d} = (4,4) \in \mathbb{Z}^2$, $L_H = \langle (6p, 6p) \rangle$, $L_K = \langle (0,0) \rangle$, for some $p \in \mathbb{Z}$, $p \neq 0$.

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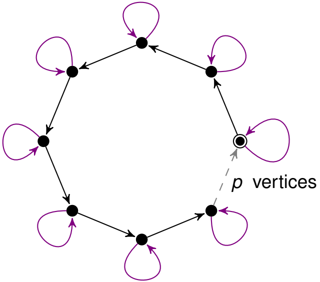

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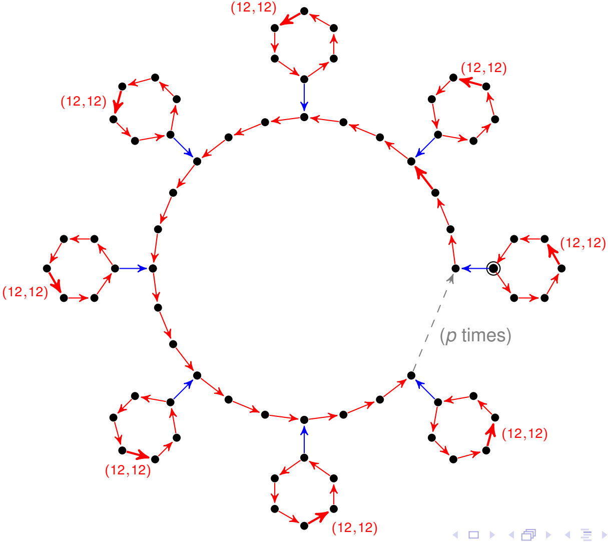

# Intersection example: case 4

Therefore, $H_p = \langle t^{(6p,6p)},\ x^3 t^{(6,6)},\ yx \rangle$, $K = \langle x^2 t^{(4,4)},\ yxy^{-1} \rangle$ and

$$H \cap K = \left\langle \begin{array}{c} (yx^{-3}y^{-1})^{-1} x^6 (yx^{-3}y^{-1}) t^{(12,12)} \\ (yx^{-3}y^{-1})^{-2} x^6 (yx^{-3}y^{-1})^2 t^{(12,12)} \\ \vdots \\ (yx^{-3}y^{-1})^{-(p-1)} x^6 (yx^{-3}y^{-1})^{p-1} t^{(12,12)} \\ yx^{3p}y^{-1} \end{array} \right\rangle$$

Note that $r(H_p) = 3$, $r(K) = 2$, but

$$r(H_p \cap K) = p + 1.$$

So, no Hanna Neumann type inequality $\tilde{r}(H \cap K) \leqslant C \cdot \tilde{r}(H) \cdot \tilde{r}(K)$ is possible in these groups.

THANKS